

# A microlocal category associated to a symplectic manifold

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## 1. INTRODUCTION

There are several ways to construct a category which is an invariant of a symplectic manifold. One is due to Fukaya and is based on Floer cohomology [12], [13]. A connection between the Fukaya theory and theory of constructible sheaves was established by Nadler and Zaslow [36], [35]. Another construction of a category starting from a symplectic manifold was carried out by Tamarkin [46] and [47]. It is based on microlocal theory of sheaves on manifolds developed by Kashiwara and Schapira in [27].

In this paper we describe yet another construction. It is based on microlocal objects, as [46] and [47] are. But instead of microlocal theory of sheaves we use asymptotics of pseudodifferential operators and Lagrangian distributions [18], [20], or rather their algebraic version described by deformation quantization [1], [9], [37], [38].

### 1.1. Motivation from Morse theory.

**1.1.1. The classical Morse filtration.** First recall that, given a function  $f$  on a  $C^\infty$  manifold  $X$ , one can study De Rham cohomology of  $X$  using a filtration of the sheaf  $\mathbb{C}_X$  by subsheaves  $\mathbb{C}_{X,t} = \mathbb{C}_{\{f(x) \geq t\}}$  for any real  $t$ . If  $f$  is a Morse function, the cohomology  $H^\bullet(X, \mathbb{C}_{X,t}/\mathbb{C}_{X,t'})$  is described in terms of critical points of  $f$ .

**1.1.2. The filtered local system of  $\mathbb{K}$ -modules.** The above can be interpreted as follows. Let

$$\Lambda = \left\{ \sum_{k=0}^{\infty} a_k \exp\left(\frac{1}{i\hbar} c_k\right) \mid a_k \in \mathbb{C}; c_k \geq 0; c_k \rightarrow \infty \right\}$$

be the Novikov ring. Let  $\mathbb{K}$  be its field of quotients which is defined the same way as  $\Lambda$ , with the condition  $c_k \geq 0$  replaced by  $c_k \in \mathbb{R}$ . Consider the trivial  $\mathbb{K}$ -module of rank one and the corresponding constant sheaf  $\mathbb{K}_X$  on  $X$ . Given a function  $f$ , consider the action of the fundamental groupoid  $\pi_1(X)$  on  $\mathbb{K}_X$  such that any class of a path  $x_0 \rightarrow x_1$  acts by multiplication by  $\exp(\frac{1}{i\hbar}(f(x_0) - f(x_1)))$ .

For any real number  $t$ , denote by  $C_{\Lambda^t, X}^\infty$  the sheaf associated to the presheaf of formal expressions

$$(1.1.1) \quad \left\{ \sum_{k=0}^{\infty} a_k \exp\left(\frac{1}{i\hbar} \varphi_k\right) \mid a_k \in C_X^\infty((\hbar)); \varphi_k \in C_X^\infty; \varphi_k \geq t; \varphi_k \rightarrow \infty \right\}$$

Define  $C_{\mathbb{K},X}^\infty$  the same way but without the condition  $\varphi_k \geq t$ . When  $t = 0$ , we denote  $C_{\Lambda^t,X}^\infty$  by  $C_{\Lambda,X}^\infty$ .

The fundamental groupoid  $\pi_1(X)$  acts on  $C_{\mathbb{K},X}^\infty$  (the simple exact meaning of this statement is explained in Definition 6.18). Horizontal sections are of the form

$$(1.1.2) \quad \sum_k a_k \exp\left(\frac{1}{i\hbar}(c_k + f(x))\right)$$

where  $a_k \in \mathbb{C}((\hbar))$ ,  $c_k \in \mathbb{R}$ , and  $c_k \rightarrow \infty$ . Now consider the sheaf  $\mathcal{F}^t(f)$  of horizontal sections that are in  $C_{\Lambda^t,X}^\infty$ . Note that  $\exp(\frac{1}{i\hbar}(c + f))$  is in  $\mathcal{F}^t(f)$  on an open set if and only if  $c \geq t - f$  on this open set. Therefore

$$(1.1.3) \quad H^\bullet(X, \mathcal{F}^t(f)) = \widehat{\oplus}_c H^\bullet(U_{c,t})((\hbar))$$

where  $U_{c,t}$  is the biggest open subset on which  $c \geq t - f$ . We see that this cohomology essentially contains all the information about the cohomology of  $X_t$  for various  $t$ . The symbol  $\widehat{\oplus}$  denotes the completed direct sum, *i.e.* the space of infinite sums

$$(1.1.4) \quad \sum_{k=1}^{\infty} A_k, \quad A_k \in H^\bullet(U_{c_k,t})((\hbar)), \quad c_k \rightarrow \infty$$

**1.1.3. The twisted De Rham complex.** The language of local systems and of actions of the fundamental groupoid makes it natural to look at flat connections.

**Definition 1.1.** Denote by  $\Omega_{\mathbb{K},X}^\bullet$ , resp.  $\Omega_{\Lambda^t,X}^\bullet$ , resp.  $\Omega_{\Lambda,X}^\bullet$ , the sheaf of differential forms with coefficients in  $C_{\mathbb{K},X}^\infty$ , resp.  $C_{\Lambda^t,X}^\infty$ , resp.  $C_{\Lambda,X}^\infty$ .

Consider the twisted De Rham complex

$$(1.1.5) \quad (\Omega_{\mathbb{K},X}^\bullet, i\hbar d_{\text{DR}} + df \wedge)$$

This complex is filtered by subcomplexes  $\Omega_{\Lambda^t,X}^\bullet$ . The fundamental groupoid acts on it preserving the differential (again, see Definition 6.18 for the exact meaning of this).

Now, for traditional local systems of finite dimensional vector spaces, locally, the cohomology of the De Rham complex is the same as the space of horizontal sections. The latter is (again, locally) the same as the derived space of horizontal sections, which is by definition the cohomology of the fundamental groupoid with coefficients in functions. In the context of  $C_{\mathbb{K},M}^\infty$ -valued forms, the first of these statements is false. In fact, the cohomology of the complex (1.1.5) is huge: regardless of  $f$ , it is the sum of cohomologies of  $d_{\text{DR}} + d\varphi \wedge$  for *all*  $\varphi$ . But if we consider the local double complex of cochains of the fundamental groupoid with coefficients in (1.1.5), we get the cohomology isomorphic to  $\mathbb{K}$ . This is easy to see. In fact, we can replace  $f$  by 0 in (1.1.5), since the two complexes are isomorphic by means of multiplication

by  $\exp(\frac{1}{i\hbar}f)$ . The value of the local double complex on a coordinate chart  $U$  becomes

$$\mathcal{C}^{p,q} = \Omega_{\mathbb{K}}^p(U^{q+1})$$

for  $p, q \geq 0$ . There are two differentials: one is  $d_{\text{DR}} : \mathcal{C}^{p,q} \rightarrow \mathcal{C}^{p+1,q}$ , the other is  $\delta : \mathcal{C}^{p,q} \rightarrow \mathcal{C}^{p,q+1}$  where for  $\omega \in \mathcal{C}^{p,q}$

$$(1.1.6) \quad \delta\omega = \sum_{j=0}^q (-1)^j p_j^* \omega$$

Here  $p_j$  is the projection  $X^{q+1} \rightarrow X^q$  along the  $j$ th factor. But the differential  $\delta$  admits a contracting homotopy

$$(1.1.7) \quad h\omega = i_0^* \omega$$

where  $i_0(x_0, \dots, x_{q-1}) = (0, x_0, \dots, x_{q-1})$ . More precisely,  $[\delta, h] = \text{Id} - r_0$  where  $r_0 = 0$  for  $q > 0$  or  $p > 0$ , and  $r_0 a = a(0)$  for  $p = q = 0$ .

The sheaf associated to the presheaf of local complexes  $\mathcal{C}^{\bullet, \bullet}$  inherits the action of the fundamental groupoid. The easiest way to express this is to say that, if

$$(1.1.8) \quad \mathcal{C}_x^{p,q} = \varinjlim_{x \in U} \mathcal{C}^{p,q}(U),$$

then there are operators

$$(1.1.9) \quad \pi_1(x, y) \times \mathcal{C}_y^{p,q} \rightarrow \mathcal{C}_x^{p,q}$$

that define an action. In a more general situation, when we start with a differential graded module  $\mathcal{E}^{\bullet}$  over  $\Omega_{\mathbb{K}, X}^{\bullet}$  with a compatible action of  $\pi_1(X)$ , they define an  $A_{\infty}$  action. This is more or less the same for all practical purposes (*cf.* 6.1).

We summarize the above as follows. Starting from a function  $f$  we constructed a filtered differential graded module  $\mathcal{E}^{\bullet}$  over  $\Omega_{\mathbb{K}, X}^{\bullet}$  with a compatible action of  $\pi_1(X)$ , namely the twisted De Rham complex (1.1.5). From that we passed to a filtered  $\mathbb{K}_X$ -module with an (*a priori*  $A_{\infty}$ ) action of  $\pi_1(X)$ . It is natural to call such an object a filtered infinity local system of  $\mathbb{K}$ -modules. (Note that the complex is filtered but  $\pi_1$  does not preserve the filtration). The goal of this paper is to generalize large parts of the above in the way that we explain next.

## 1.2. Lagrangian submanifolds.

**1.2.1. Review of the results.** Let  $M$  be a symplectic manifold and  $L_0, L_1$  its Lagrangian submanifolds. Under some topological assumptions that we will list below, we will construct an infinity-local system of  $\mathbb{K}$ -modules  $\mathcal{C}^{\bullet}(L_0, L_1)$  on  $M$ . In examples, this infinity local system is often filtered. The precise topological conditions that guarantee it being filtered are yet to be determined. Complexes  $\mathcal{C}^{\bullet}(L_0, L_1)$  have a structure of an  $A_{\infty}$ -category enriched in  $A_{\infty}$  local systems of  $\mathbb{K}$ -modules (we will develop this in detail in

a subsequent work). When  $M = T^*X$ ,  $L_0 = \text{graph}(0)$ , and  $L_1 = \text{graph}(df)$ , we recover the construction we discussed above (with some modification).

The topological conditions, most probably much too conservative for large parts of the construction, are as follows.

1) The manifold  $M$  has an  $\text{Sp}^4$ -structure (cf. 12.3). In other words, for an almost complex structure compatible with  $\omega$ , consider the first Chern class  $c_1(M)$  of the tangent bundle viewed as a complex vector bundle. Then  $2c_1(M)$  must be trivial in  $H^2(M, \mathbb{Z}/4\mathbb{Z})$ . An  $\text{Sp}^4$  structure is a trivialization of  $2c_1(M)$ .

2) The image of the pairing of the class of the symplectic form with the image of the Hurewicz morphism is zero:  $\langle \pi_2(M), [\omega] \rangle = 0$ .

(The properties of Lagrangian submanifolds that are usually considered in Fukaya theory, such as exactness, grading, and existence of a Spin structure, all make their appearance in our considerations, as well as in [47]. Their exact role will be discussed in a subsequent work).

The infinity local system will be constructed in several steps indicated below. The meaning of all the terms used will be explained later in the introduction and/or in the rest of the article. All steps are possible under some additional conditions.

a) We will introduce a sheaf of algebras  $\mathcal{A}_M$  with a flat connection on  $M$ . On this sheaf, the fundamental groupoid  $\pi_1(M)$  will act up to inner automorphisms. Denote by  $\mathcal{A}_M^\bullet$  the differential graded algebra of  $\mathcal{A}_M$ -valued forms, with the differential given by the connection.

b) Consider two modules  $\mathcal{V}$  and  $\mathcal{W}$  over  $\mathcal{A}_M$  with a compatible action of  $\pi_1(M)$  and a compatible connection. Denote by  $\mathcal{V}^\bullet, \mathcal{W}^\bullet$  the differential graded modules of forms with values in  $\mathcal{V}$  or  $\mathcal{W}$ . Then the standard complex computing their Ext over  $\mathcal{A}_M^\bullet$  has a structure of a  $\Omega_{\mathbb{K}, M}^\bullet$ -module with a (twisted)  $A_\infty$  action of  $\pi_1(M)$ .

c) Given an  $\Omega_{\mathbb{K}, M}^\bullet$ -module with a (twisted)  $A_\infty$  action of  $\pi_1(M)$ , we will construct an infinity local system as in (1.1.8).

d) To construct modules  $\mathcal{V}$  as in b), note that we can start with an  $\mathcal{A}_M$ -module with a compatible connection and a compatible action of a bigger groupoid  $\tilde{\mathbf{G}}_M$  that maps onto  $\pi_1(M)$  in such a way that the kernel of this map acts by inner automorphisms.

e) Given a Lagrangian submanifold  $L$ , we notice that there exists a subgroupoid of  $\tilde{\mathbf{G}}_M|L$  on  $L$ , as well as an  $\mathcal{A}_M|L$ -module with a compatible connection and a compatible action of this subgroupoid. Now we can get an object as in d) by an induction procedure.

We will now outline the steps a)-e) in more detail.

### 1.3. Deformation quantization.

**1.3.1. The twisted De Rham complex, deformation quantization, and Ext functors.** The fact that the twisted De Rham complex can be interpreted in terms of homological algebra had been known for a long time.

Namely, let  $\mathcal{D}_\hbar(X)$  be the ring of  $C^\infty$   $\hbar$ -differential operators, *i.e.* the subalgebra of all differential operators which is generated, in any local coordinate system, by  $F(x_1, \dots, x_n)$  for all functions  $F$  and by  $i\hbar \frac{\partial}{\partial x_j}$  for all  $j$ . Here  $\hbar$  can be any nonzero number, but it is easy to modify this construction to make  $\hbar$  a formal parameter (in which case  $\mathcal{D}_\hbar(X)$  is the Rees ring [5]). The algebra  $\mathcal{D}_\hbar(X)$  acts on the space of functions on  $X$ . Denote the corresponding module by  $V_0$ . Now note that a function  $f$  defines an automorphism of  $\mathcal{D}_\hbar(X)$ , namely the conjugation with  $\exp(\frac{1}{i\hbar}f)$ . When  $\hbar$  is not a number but a formal parameter, it is not clear how to define  $\exp(\frac{1}{i\hbar}f)$  but conjugation by it makes perfect sense. Namely, in any coordinate system it sends  $F(x_1, \dots, x_n)$  to itself for all  $F$  and  $i\hbar \frac{\partial}{\partial x_j}$  to  $i\hbar \frac{\partial}{\partial x_j} + \frac{\partial f}{\partial x_j}$  for all  $j$ . It can be easily shown that  $\text{Ext}_{\mathcal{D}_\hbar}^\bullet(V_0, V_f)$  can be computed by the twisted De Rham complex. When  $\hbar$  is a nonzero number, this complex is of course isomorphic to the standard De Rham complex. When  $\hbar$  is a formal parameter, this complex is

$$(1.3.1) \quad (\Omega^\bullet(X)[\hbar], i\hbar d_{\text{DR}} + df \wedge)$$

When we formally invert  $\hbar$  the cohomology of this differential becomes easier to compute because we can use the spectral sequence associated to the filtration by powers of  $\hbar$ . The first differential in this spectral sequence is  $df \wedge$ . When  $f$  has isolated nondegenerate critical points, the cohomology of this differential, and therefore the cohomology of the twisted De Rham complex, is concentrated in the top degree  $n$  and its dimension over the field  $\mathbb{C}((\hbar))$  of Laurent series is equal to the number of critical points.

Now let  $\mathbb{A}_M$  be a deformation quantization of  $C^\infty(M)$  (*cf.* [1]; we recall the definitions in 3.2). When  $M = T^*X$ , there is the canonical deformation quantization that is a certain completion of  $\mathcal{D}_\hbar(X)$ . (Another, arguably more correct, deformation is a completion of the algebra of  $\hbar$ -differential operators on half-forms). The algebra  $\mathbb{A}_M$  is a reasonable replacement of  $\mathcal{D}_\hbar(X)$ , although it is no longer an algebra over  $\mathbb{C}[\hbar]$  but only over  $\mathbb{C}[[\hbar]]$ . In particular it does not allow any specialization at a nonzero number  $\hbar$ .

In mid-eighties, Feigin suggested an idea based on the intuition from algebraic theory of  $\mathcal{D}$ -modules [5]. According to this idea, and to a subsequent work [6] of Bressler and Soibelman, one should associate to a Lagrangian submanifold  $L$  a sheaf of  $\mathbb{A}_M$ -modules  $\mathbb{V}_L$  supported on  $L$ . Then  $\text{Ext}^\bullet(\mathbb{V}_{L_0}, \mathbb{V}_{L_1})$  should somehow be a first approximation for a more interesting theory, namely the Floer cohomology. The latter also sees intersection points of transversal Lagrangian submanifolds, but in a much subtler way. Those intersection points define cochains (not necessarily cocycles) of the Floer complex that are not of the same but of different degrees (given by the Maslov index). Furthermore, the differential in the Floer complex may send one such cochain to a linear combination of other points (in other words, there may be instanton corrections). The standard homological algebra seems to be unable to catch these effects.

Below we will outline several tools that, combined, seem to allow to construct a category some (but not all) of whose objects come from Lagrangian submanifolds and which is much closer to the Fukaya category than the bare category of  $\mathbb{A}_M$ -modules.

**1.3.2. The Fedosov construction.** The work of Fedosov [9] provided a simple and very efficient tool for working with deformation quantization of symplectic manifolds. Recall that a local model for deformation quantization is the Weyl algebra  $C^\infty(M)[[\hbar]]$  with the Moyal-Weyl product  $*$ . The key properties of this product are that it is  $\mathrm{Sp}(2n, \mathbb{R})$ -invariant and that

$$[\xi_j, x_k] = i\hbar\delta_{jk}; [x_j, x_k] = [\xi_j, \xi_k] = 0.$$

The local model for the Fedosov construction is as follows. Start with the space  $\widehat{\mathbb{A}}$  of power series in formal variables  $\widehat{x}_j$ ,  $\widehat{\xi}_j$ , and  $\hbar$ ,  $1 \leq j \leq n$ . Turn it into an algebra by introducing the Moyal-Weyl product. Now consider the algebra of  $\widehat{\mathbb{A}}$ -valued differential forms on the Darboux chart with coordinates  $x_j, \xi_j$ . This algebra is equipped with the differential given by formula

$$(1.3.2) \quad \nabla_{\mathbb{A}} = \sum_{j=1}^n \left( \left( \frac{\partial}{\partial x_j} - \frac{\partial}{\partial \widehat{x}_j} \right) dx_j + \left( \frac{\partial}{\partial \xi_j} - \frac{\partial}{\partial \widehat{\xi}_j} \right) d\xi_j \right)$$

(*cf.* also (3.1.1)). The cohomology algebra of this differential is the usual deformation quantization.

For a general symplectic manifold  $M$ , one replaces a deformation  $\mathbb{A}_M$  with the algebra  $\Omega^\bullet(M, \widehat{\mathbb{A}}_M)$  of  $\widehat{\mathbb{A}}_M$ -valued differential forms on  $M$ . Here  $\widehat{\mathbb{A}}_M$  is the bundle of algebras with fiber  $\widehat{\mathbb{A}}$ . The differential on the algebra  $\Omega^\bullet(M, \widehat{\mathbb{A}}_M)$  is a chosen Fedosov connection. On any local Darboux chart, this algebra is isomorphic to the one discussed in the previous paragraph.

Note that the usual intuition about flat connections does not work here. Namely, there is no action of the fundamental groupoid (monodromy) preserving this flat connection. In fact, even locally, the algebra of horizontal sections is not at all isomorphic to the fiber. This feature will change rather radically after a modification that we introduce next. Much of what follows is based on the idea suggested to the author by Alexander Karabegov: extend the work of Fedosov so that it will describe an asymptotic version of Maslov's theory of canonical operators and of Hörmander's theory of Lagrangian distributions (*cf.* [18], [20], [32]). Actually, the constructions below require nothing but a systematic introduction into deformation quantization of quantities of the form (1.3.3) below. They do however have very strong connections to [18], [20], [32]. We discuss these connections in Appendices (Sections 12, 13, 15, and 17). Note that exponentials (1.3.3) were considered in deformation quantization since the introduction of the subject, in particular in [1], in [7] in [10].

**1.3.3. The extended Fedosov construction.** Let us start with a remark about what happens when one tries systematically to introduce into deformation quantization quantities of the form

$$(1.3.3) \quad \exp\left(\frac{1}{i\hbar}\varphi\right).$$

Let us do this at the level of the algebra of formal series  $\widehat{\mathbb{A}}$ . All such quantities where  $\varphi$  are power series starting with cubic terms become elements of a new algebra automatically as soon as one replaces  $\widehat{\mathbb{A}}$  by a completion  $\widehat{\widehat{\mathbb{A}}}$  (cf. 4.1). We interpret quantities (1.3.3) where  $\varphi$  are quadratic as elements of the 4-fold covering group  $\mathrm{Sp}^4(2n, \mathbb{R})$  (see the remark below). To add elements (1.3.3) where  $\varphi$  is constant, we tensor our algebra by the Novikov field  $\mathbb{K}$  (as in 1.1.2).

*Remark 1.2.* Here is an explanation of the presence of  $\mathrm{Sp}^4$  (cf. section 12 for definitions). The Lie algebra of derivations of the algebra  $\widehat{\mathbb{A}}$  has a subalgebra consisting of elements  $\frac{1}{i\hbar}\mathrm{ad}(q(\widehat{x}, \widehat{\xi}))$  where  $q$  is a quadratic function. This Lie subalgebra is isomorphic to  $\mathfrak{sp}(2n)$ , and its action is the standard action by linear coordinate changes. Consider the  $\widehat{\mathbb{A}}$ -module  $\mathbb{C}[[\widehat{x}, \hbar]][\hbar^{-1}]$  on which  $\widehat{x}$  acts by multiplication and  $\widehat{\xi}$  by  $i\hbar\frac{\partial}{\partial \widehat{x}}$ . On it,  $\frac{1}{i\hbar}\widehat{x}_j\widehat{\xi}_k$  acts by  $\widehat{x}_j\frac{\partial}{\partial \widehat{x}_k} + \frac{1}{2}\delta_{jk}$ . Note that  $\mathrm{ad}(\frac{1}{i\hbar}\widehat{x}_j\widehat{\xi}_k)$  form a basis of the subalgebra  $\mathfrak{gl}(n)$  inside  $\mathfrak{sp}(2n)$ . We see that one can integrate the action of this Lie subalgebra on the module to an action of the group, put the most natural way to do this is to pass to the two-fold cover  $\mathrm{ML}(n, \mathbb{R})$  consisting of pairs  $\{(g, \zeta) \mid \det(g) = \zeta^2\}$ . One cannot extend this group action to the full symplectic group. To achieve that, we will have to extend the module considerably. But the group containing  $\mathrm{ML}(n)$  is not  $\mathrm{Sp}(2n)$  but its universal two-fold cover  $\mathrm{Mp}(2n)$ . The group  $\mathrm{Sp}^4$  contains  $\mathrm{Mp}(2n)$  as a normal subgroup with quotient  $\mathbb{Z}/2\mathbb{Z}$ . We pass to this bigger group because it behaves better with respect to Lagrangian subspaces. For example, if a symplectic manifold  $M$  has a real polarization, then  $M$  has an  $\mathrm{Sp}^4(2n)$ -structure but not necessarily an  $\mathrm{Mp}(2n)$ -structure. On a more basic level, the pre-image of  $\mathrm{GL}(n, \mathbb{R})$  in  $\mathrm{Sp}^4(2n, \mathbb{R})$  splits, i.e. is isomorphic to  $\mathrm{GL}(n, \mathbb{R}) \times \mathbb{Z}/4\mathbb{Z}$ .

Finally, we do not add elements (1.3.3) where  $\varphi$  are linear, for the following reason. Note that  $\mathrm{ad}(\frac{1}{i\hbar}\widehat{\xi}_j) = \frac{\partial}{\partial \widehat{x}_j}$  and  $\mathrm{ad}(\frac{1}{i\hbar}\widehat{x}_j) = -\frac{\partial}{\partial \widehat{\xi}_j}$ . Exponentials of these operators should be shifts in formal variables  $\widehat{x}_j$  and  $\widehat{\xi}_j$ . But such shifts do not act on power series. Instead, they should correspond to shifts acting from one fiber of the associated bundle of algebras to another. These shifts will be discussed in 1.4 below. One does not need to add them, they will act automatically as long as topological conditions 1), 2) from 1.2.1 are satisfied.

We get an algebra  $\mathcal{A}$  containing  $\widehat{\widehat{\mathbb{A}}}$ ,  $\mathbb{C}[\mathrm{Sp}^4(2n)]$ , and  $\mathbb{K}$  as subalgebras. The associated bundle of algebras  $\mathcal{A}_M$  carries a Fedosov connection  $\nabla_{\mathcal{A}}$

that extends the one on  $\widehat{\mathbb{A}}_M$ . For all we know, the cohomology of the De Rham complex of this connection is huge. But the bundle of algebras  $\mathcal{A}_M$  carries another structure that we are going to discuss next.

**1.4. The action of  $\pi_1$  up to inner automorphisms.** It turns out that, if conditions 1) and 2) from 1.2.1 are satisfied, the fundamental groupoid  $\pi_1(M)$  acts on the bundle of algebras  $\mathcal{A}_M$  up to inner automorphisms. The notion of such an action is defined in section 5. Moreover, the Fedosov connection  $\nabla_{\mathcal{A}}$  extends to a flat connection up to inner derivations compatible with this action (cf. 5.7.2).

All the requisite notions are well-known and go back to Grothendieck. The version that suits our purposes is developed here in section 5. For the readers convenience we introduce these notions gradually, starting with the case of a group acting on an algebra, though the generality we need is that of a Lie groupoid acting on a sheaf of algebras. The Lie groupoid in question will be the fundamental groupoid or its extension by a bundle of Lie groups.

**1.5. From an action up to inner automorphisms to an  $A_\infty$  local system.** In section 6 we explain that, given an action of a groupoid  $\mathcal{G}$  on a sheaf of algebras  $\mathcal{A}$  up to inner automorphisms and given two  $\mathcal{A}$ -modules  $\mathcal{V}$  and  $\mathcal{W}$  with a compatible action of the groupoid, the standard complex  $\mathcal{C}^\bullet(\mathcal{V}, \mathcal{A}, \mathcal{W})$  that computes  $\text{Ext}_{\mathcal{A}}^\bullet(\mathcal{V}, \mathcal{W})$  carries a (*twisted*)  $A_\infty$  action of  $\mathcal{G}$ . We make a similar argument when  $\mathcal{A}$  carries a flat connection up to inner derivations. (Twisted  $A_\infty$  actions are discussed in 16. They are needed because the action in 1.4 is continuous only locally).

Let  $\mathcal{A}_M^\bullet$  be the sheaf of  $\mathcal{A}_M$ -valued forms on  $M$ . The above procedure starts with two differential graded  $\mathcal{A}$ -modules  $\mathcal{V}^\bullet, \mathcal{W}^\bullet$  with compatible actions of  $\pi_1(M)$  and produces the standard complex  $\mathcal{C}^\bullet(\mathcal{V}^\bullet, \mathcal{A}^\bullet, \mathcal{W}^\bullet)$  which is a sheaf of  $\Omega_{\mathbb{K}, M}^\bullet$ -modules with a compatible *twisted*  $A_\infty$  action of  $\pi_1(M)$ . Finally, for an open chart  $U$  in  $M$ , consider the double complex  $\mathcal{C}^{\bullet, \bullet}(\mathcal{V}^\bullet, \mathcal{W}^\bullet)(U)$  where  $\mathcal{C}^{p, q}(U)$  is the space of  $q$ -cochains of  $\pi_1(U)$  with coefficients in the graded component  $\mathcal{C}^p(\mathcal{V}^\bullet, \mathcal{A}^\bullet, \mathcal{W}^\bullet)$ , as in the second part of 1.1.3. Let

$$\mathcal{C}_x^{\bullet, \bullet} = \varinjlim_{x \in U} \mathcal{C}^{\bullet, \bullet}(\mathcal{V}^\bullet, \mathcal{W}^\bullet)(U)$$

be the stalk at a point  $x$ . As we indicated in 1.1.3 (after (1.1.8)), these complexes form an  $A_\infty$  local system of  $\mathbb{K}$ -modules. We denote this local system by  $\mathbb{R}\text{HOM}(\mathcal{V}^\bullet, \mathcal{W}^\bullet)$ .

We sum up the construction up to this point in section 8.

**1.6. Objects constructed from Lagrangian submanifolds.** . We proceed to construct a differential graded module  $\mathcal{V}_L$  as in 1.5 starting from a Lagrangian submanifold  $L$ . This is done using an induction procedure that is explained in Section 9, in particular in 9.2. In Section 10, we prove that the general construction, when applied to  $M = \mathbb{R}^{2n}$ ,  $L_0 = \text{graph}(0)$ , and  $L_1 = \text{graph}(df)$ , reproduces the one in 1.1.2, with the one important distinction. Namely, the filtered  $A_\infty$  local system  $\mathbb{R}\text{HOM}(\mathcal{V}_{L_0}^\bullet, \mathcal{V}_{L_1}^\bullet)$  whose



construction is outlined above is a module over a trivial local system of differential graded algebras whose fiber is the algebra

$$(1.6.1) \quad \mathcal{S}^\bullet = C^\bullet(\text{MPar}(n), \mathbb{K})$$

of cochains of the group  $\text{MPar}(n)$  with coefficients in the Novikov field  $\mathbb{K}$ . Here  $\text{MPar}(n)$  is the parabolic subgroup of the group  $\text{Sp}^4(2n)$  which is the pre-image of the stabilizer of the Lagrangian submanifold  $\xi_1 = \dots = \xi_n = 0$  in  $\text{Sp}(2n)$ . We prove that the general construction outlined in 1.5 is the tensor product of  $\mathcal{S}^\bullet$  by the filtered local system described in 1.1.2.

*Remark 1.3.* There probably exists a correct way of factoring out the maximal ideal of  $\mathcal{S}^\bullet$  and in particular recovering the exact answer as in 1.1.2. Note that the algebra  $\mathcal{S}^\bullet$  plays a vital role in the computation in section 10. Namely, the vanishing of the cohomology of  $\text{MPar}(n)$  with coefficients in a certain class of modules leads to a vanishing result for all components involving a factor  $\exp(\frac{1}{i\hbar}\varphi(x, \hat{x}))$  where the quadratic part of  $\varphi$  with respect to  $\hat{x}$  is nonzero. Cf. Lemma 10.5, Corollary 10.6 (which we interpret as stationary phase statements of some sort).

**1.6.1. The example of a two-dimensional torus.** In 11, we compute  $\mathbb{R}\text{HOM}(\mathcal{V}_{L_0}^\bullet, \mathcal{V}_{L_m}^\bullet)$  where  $M = \mathbb{R}^2/\mathbb{Z}^2$ ,  $L_0 = \{\xi = 0\}$ , and  $L_m = \{\xi = mx\}$ . The answer is the trivial bundle whose fiber is the space of matrices indexed by  $k, \ell \in \mathbb{Z}$  with coefficients in  $\mathcal{S}^\bullet$ . If  $\gamma_1, \gamma_2$  are the two generators of the fundamental group  $\pi_1(M) \xrightarrow{\sim} \mathbb{Z}^2$ , then the action of  $\pi_1(M)$  on the matrix units  $\mathbf{E}_{k\ell}$  is given by

$$\gamma_1^q \gamma_2^p : \mathbf{E}_{k\ell} \mapsto \exp\left(\frac{1}{i\hbar}\left(\frac{mq^2}{2} + q(\ell - k)\right)\right) \mathbf{E}_{k+p, \ell+p-mq}$$

As a consequence (Corollary 11.3), horizontal sections of this local system have the same algebraic expression as theta functions. This agrees with the computation of the Fukaya category of  $M$  given by Polishchuk and Zaslow in [41].

## 1.7. Microlocal category of sheaves.

**1.7.1. The microlocal category of Tamarkin.** In [46], Tamarkin defined the category  $D(T^*X)$  for a manifold  $X$ . This is a full subcategory of the differential graded category of complexes of sheaves on  $X \times \mathbb{R}$ . Below are the key properties of the differential graded category  $D(T^*X)$ .

(1) For  $c \geq 0$ , there is a natural transformation  $\tau_c : \text{Id} \rightarrow (T_c)_*$  where, for  $(x, t) \in X \times \mathbb{R}$ ,  $T_c(x, t) = (x, t + c)$ . One has  $\tau_c \tau_{c'} = \tau_{c+c'}$ . Define

$$\text{HOM}(\mathcal{F}, \mathcal{G}) = \prod'_{c \geq 0} \mathbb{R}\text{Hom}(\mathcal{F}, (T_c)_* \mathcal{G})$$

where  $\prod'$  is the subset of the direct product consisting of all elements  $(v_c)$  such that  $v_c = 0$  for all but countably many  $c_k, k = 1, 2, \dots$ , satisfying

$c_k \rightarrow \infty$ . Then  $\mathrm{HOM}(\mathcal{F}, \mathcal{G})$  is a complex of modules over the Novikov ring  $\Lambda_{\mathbb{Z}} = \{\sum_{k=0}^{\infty} a_k e^{-\frac{c_k}{i\hbar}}\}$  where  $a_k \in \mathbb{Z}$ ,  $c_k \in \mathbb{R}$ ,  $c_k \geq 0$ , and  $c_k \rightarrow \infty$ .

*Remark 1.4.* For a general sheaf  $\mathcal{F}$  there is no relation between its behavior on an open subset  $U$  and on the shift of  $U$  by  $c$  in the  $t$  direction. But Tamarkin's subcategory has a remarkable property that the natural transformation  $\tau_c$  exists. A key example is provided by sheaves  $\mathcal{F}_f$  defined in the paragraph below.

(2) For every object  $\mathcal{F}$  of  $D(T^*X)$ , a closed subset  $\mu S(\mathcal{F})$  is defined, called the microsupport of  $\mathcal{F}$ . Let  $f$  be a smooth function on  $X$ . Denote  $\mathcal{F}_f = \mathbb{Z}_{\{t+f(x) \geq 0\}}$ . Then  $\mu S(\mathcal{F}_f) = \mathrm{graph}(df)$ . (Observe that  $T_c^* \mathcal{F}_f = \mathcal{F}_{f-c}$ ; the morphism  $\tau_c : \mathcal{F} \rightarrow T_{c*} \mathcal{F}$  is the restriction to the subset  $\{t - f - c \geq 0\}$  of  $\mathbb{Z}_{\{t-f \geq 0\}}$ ).

(3) For a Morse function  $f$ , the complex  $\mathrm{HOM}(\mathcal{F}_0, \mathcal{F}_f)$  is quasi-isomorphic to the Morse complex of  $f$ .

(4) Let  $\mathbf{T}^2$  be the standard 2-torus with the flat symplectic structure. One defines the category  $D(\mathbf{T}^2)$  of objects of  $D(T^*\mathbb{R}^1)$  equivariant under certain projective action of  $\mathbb{Z}^2$ . For every Lagrangian submanifold of  $\mathbf{T}^2$  of the form  $a\xi + bx = c$ ,  $a, b, c$  being integers, one constructs an object  $\mathcal{F}_{a,b,c}$  of  $D(\mathbf{T}^2)$ . The full subcategory generated by these objects is isomorphic to the full subcategory of the Fukaya category generated by Lagrangian submanifolds  $a\xi + bx = c$  as computed by Polishchuk-Zaslow in [41].

*Remark 1.5.* The category  $D(\mathbf{T}^2)$  can be defined either as a partial case of the general construction [47] or by an explicit procedure that we recall in 11.3.

(5) *Theorem B.* Let  $\Phi$  be a Hamiltonian symplectomorphism of  $T^*X$  which is equal to identity outside a compact subset. There exists a functor  $T_{\Phi} : D(T^*X) \rightarrow D(T^*X)$  such that, if  $\mu S(\mathcal{F})$  is compact,  $\mu S(T_{\Phi}(\mathcal{F})) \subset \Phi(\mu S(\mathcal{F}))$ . For every  $\mathcal{F}$  and  $\mathcal{G}$ ,  $\mathrm{HOM}(\mathcal{F}, \mathcal{G})$  and  $\mathrm{HOM}(\mathcal{F}, T_{\Phi}(\mathcal{G}))$  are isomorphic modulo  $\Lambda_{\mathbb{Z}}$ -torsion. Similarly for  $\mathrm{HOM}(\mathcal{F}, \mathcal{G})$  and  $\mathrm{HOM}(T_{\Phi}(\mathcal{F}), \mathcal{G})$ .

(6) *Theorem A.* Let  $\mathcal{F}$  and  $\mathcal{G}$  be objects of  $D(T^*X)$  such that  $\mu S(\mathcal{F})$  and  $\mu S(\mathcal{G})$  are compact and do not intersect. Then  $\mathrm{HOM}(\mathcal{F}, \mathcal{G}) = 0$  modulo  $\Lambda_{\mathbb{Z}}$ -torsion.

For the sake of completeness, let us indicate how some of the above constructions are carried out. For a sheaf  $\mathcal{F}$  on  $X \times \mathbb{R}$ , let  $\mathrm{SS}(\mathcal{F})$  be its singular support as defined in [27]. Let  $D(T^*X)$  be the left orthogonal complement to the subcategory of sheaves  $\mathcal{G}$  such that  $\mathrm{SS}(\mathcal{G})$  is contained in  $\{\tau \leq 0\}$ , where  $\tau$  is the variable dual to the coordinate  $t$  on  $\mathbb{R}$ . The microsupport of an object  $\mathcal{F}$  is defined by  $\mu S(\mathcal{F}) = \{(x, \xi) \in T^*X \mid (x, \xi, t, 1) \in \mathrm{SS}(\mathcal{F}) \text{ for some } t \in \mathbb{R}\}$ .

Tamarkin's current work [47] generalizes the construction of  $D(T^*X)$  to any symplectic manifold  $M$ .

**1.7.2. Comparisons between the categories.** As we can see, many properties of the category  $D(T^*X)$  are parallel to those of categories such as

$\mathcal{A}_M^\bullet$ -modules with an  $A_\infty$  action of  $\pi_1(M)$ . These include (1) (the second half), (3), and (4). Property (5) is very likely to hold. Properties (2) and (6) need further study (see next remark).

The following idea probably allows to construct a functor from  $(\mathcal{A}_M, \pi_1(M))$ -modules on  $T^*X$  satisfying some conditions to sheaves on  $X \times \mathbb{R}$ . For such a module  $\mathcal{V}^\bullet$ , assume that  $\mathbb{R}\text{HOM}(\mathcal{V}_0^\bullet, \mathcal{V}^\bullet)$  is a *filtered* infinity local system as, for example, in Conjecture 9.8 if the latter is true. Denote the filtration by  $\text{Filt}_a$ ,  $a \in \mathbb{R}$ . Then the stalk at  $(x, t)$  of the sheaf corresponding to  $\mathcal{V}^\bullet$  should be the  $\text{Filt}_t$  part of the complex that computes local cohomology of this infinity local system at  $x$ .

*Remark 1.6.* Our source of defining  $(\mathcal{A}_M, \pi_1(M))$ -modules are *oscillatory modules*. (Their original version was defined in [48]). Oscillatory modules as defined here in 8.2 are actually complexes of sheaves. It is possible to relax the definition somewhat and only require them to carry a differential  $\nabla_{\mathcal{V}}$  satisfying  $\nabla_{\mathcal{V}}^2 = \frac{1}{i\hbar}\omega$  where  $\omega$  is the symplectic form. (In other words, we can use the groupoid  $\tilde{\mathbf{G}}_M$  as defined in 7.2.1 and not in 7.2.2). If we allow this, we seem to gain much more generality. For example, it will be much easier to construct an oscillatory module not only from a Lagrangian but from a coisotropic submanifold (as discussed in [24]) and maybe for more general submanifolds. On the other hand, it seems that the condition  $\nabla_{\mathcal{V}}^2 = 0$  may be what is needed to define the microlocal support  $\mu S(\mathcal{V}^\bullet)$  (the latter should be some version of the support of the differential  $\nabla_{\mathcal{V}}$ ). *Cf.*, for example, an explicit formula for  $\nabla_{\mathcal{V}}$  given by (9.4.3).

*Remark 1.7.* Much of the motivation behind our approach came from [51]. We do not know any rigorous link between the two works. It would be very interesting to relate our methods to the study of asymptotics of eigenvalues of the Schrödinger operator.

**1.7.3. Acknowledgements.** I am grateful to Dima Tamarkin for fruitful discussions and for many explanations of his works. As already indicated above, much of the present paper originated from earlier ideas of Boris Feigin and Sasha Karabegov.

## 2. $\mathbb{R}\text{Hom}$ AND THE TWISTED DE RHAM COMPLEX

### 2.1. Deformation quantization algebra. Put

$$\mathbb{A} = C^\infty(\mathbb{R}^{2n})[[\hbar]]$$

with the Moyal-Weil product

$$(f * g)(x, \xi) = \exp\left(\frac{i\hbar}{2}\left(\frac{\partial}{\partial \xi} \frac{\partial}{\partial y} - \frac{\partial}{\partial x} \frac{\partial}{\partial \eta}\right)\right)(f(x, \xi)g(y, \eta))|_{x=y, \xi=\eta}$$

For a function  $f(x)$  denote

$$\mathbb{V}_f = \mathbb{A} / \sum_j \mathbb{A}(\xi_j - \frac{\partial f}{\partial x_j})$$

or, in a simplified notation,

$$\mathbb{V}_f = \mathbb{A}/\mathbb{A}(\xi - f'(x))$$

**Lemma 2.1.** *As a  $\mathbb{C}[[\hbar]]$ -module,  $\mathbb{V}_f$  is isomorphic to  $C^\infty(\mathbb{R}^n)[[\hbar]]$  on which  $x_j$  acts by multiplication and  $\xi_j$  by  $i\hbar \frac{\partial}{\partial x_j} + \frac{\partial f}{\partial x_j}$ .*

## 2.2. The complex computing $\mathbb{R}\mathrm{Hom}(\mathbb{V}_0, \mathbb{V}_f)$ .

**Lemma 2.2.** *The complex  $(\Omega^\bullet(\mathbb{R}^n)[[\hbar]], i\hbar d_{\mathrm{DR}} + df \wedge)$  computes  $\mathrm{Ext}_{\mathbb{A}}^\bullet(\mathbb{V}_0, \mathbb{V}_f)$*

*Proof.* Fix a basis  $e_1, \dots, e_n$  of  $\mathbb{C}^n$ . Let  $e^1, \dots, e^n$  be the dual basis of  $(\mathbb{C}^n)^*$ . Let  $\mathcal{R}_k = \mathbb{A} \otimes \wedge^k(\mathbb{C}^n)$ . If  $e_1, \dots, e_n$  is a basis of  $\mathbb{C}^n$ , define the differential

$$(2.2.1) \quad \partial(a \otimes e_{j_1} \wedge \dots \wedge e_{j_k}) = \sum_{p=1}^k (-1)^p a \xi_{j_p} \otimes e_{j_1} \wedge \dots \wedge \widehat{e_{j_p}} \wedge \dots \wedge e_{j_k}$$

The complex  $(\mathcal{R}_\bullet, \partial)$  is a free resolution of the module  $\mathbb{V}_0$ . The complex  $\mathrm{Hom}_{\mathbb{A}}(\mathcal{R}_\bullet, \mathbb{V}_f)$  becomes

$$(2.2.2) \quad C^k = \wedge^k(\mathbb{C}^n)^* \otimes \mathbb{V}_f;$$

$$(2.2.3) \quad d(e^{j_1} \wedge \dots \wedge e^{j_k} \otimes v) = \sum_{p=1}^k e^{j_1} \wedge \dots \wedge e^{j_k} \wedge e_p \otimes \xi_p v$$

which is isomorphic to  $(\Omega^\bullet(\mathbb{R}^n)[[\hbar]], i\hbar d_{\mathrm{DR}} + df \wedge)$  because of Lemma 2.1.  $\square$

## 3. THE WEYL ALGEBRA AND THE FEDOSOV CONNECTION

### 3.1. The case of $\mathbb{R}^{2n}$ . Set

$$\widehat{\mathbb{A}} = \mathbb{C}[[\widehat{x}_1, \dots, \widehat{x}_n, \widehat{\xi}_1, \dots, \widehat{\xi}_n, \hbar]]$$

with the Moyal-Weyl product

$$(f * g)(\widehat{x}, \widehat{\xi}) = \exp\left(\frac{i\hbar}{2}\left(\frac{\partial}{\partial \widehat{\xi}} \frac{\partial}{\partial \widehat{y}} - \frac{\partial}{\partial \widehat{x}} \frac{\partial}{\partial \widehat{\eta}}\right)\right)(f(\widehat{x}, \widehat{\xi})g(\widehat{y}, \widehat{\eta}))|_{\widehat{x}=\widehat{y}, \widehat{\xi}=\widehat{\eta}}$$

Define the operator on  $\widehat{\mathbb{A}}$ -valued forms by

$$(3.1.1) \quad \nabla_{\mathbb{A}} = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial \widehat{x}}\right)dx + \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \widehat{\xi}}\right)d\xi$$

This is the Fedosov connection (in the partial case of a flat space). One has  $\nabla_{\mathbb{A}}^2 = 0$ ; the complex  $(\Omega^\bullet(\mathbb{R}^{2n}, \widehat{\mathbb{A}}), \nabla_{\mathbb{A}})$  is quasi-isomorphic to  $C^\infty(\mathbb{R}^{2n})[[\hbar]]$ . The latter embeds quasi-isomorphically to the former by means of

$$(3.1.2) \quad f \mapsto f(x + \widehat{x}, \xi + \widehat{\xi}).$$

**3.1.1. Infinitesimal symmetries of the deformation quantization algebra on a formal neighborhood.** Let  $\widehat{\mathbb{A}} = \mathbb{C}[[\widehat{x}_1, \dots, \widehat{x}_n, \widehat{\xi}_1, \dots, \widehat{\xi}_n, \hbar]]$  with the Moyal-Weyl product as in 2.1. Put

$$\mathfrak{g} = \text{Der}_{\text{cont}}(\widehat{\mathbb{A}}) = \frac{1}{i\hbar} \widehat{\mathbb{A}} / \frac{1}{i\hbar} \mathbb{C}[[\hbar]]; \quad \widetilde{\mathfrak{g}} = \frac{1}{i\hbar} \widehat{\mathbb{A}}$$

viewed as Lie algebras with the bracket  $a * b - b * a$ .

Introduce the grading

$$(3.1.3) \quad |\widehat{x}_i| = |\widehat{\xi}_i| = 1; \quad |\hbar| = 2.$$

One has a central extension

$$(3.1.4) \quad 0 \rightarrow \frac{1}{i\hbar} \mathbb{C}[[\hbar]] \rightarrow \widetilde{\mathfrak{g}} \rightarrow \mathfrak{g} \rightarrow 0,$$

as well as

$$(3.1.5) \quad \mathfrak{g} = \prod_{i=-1}^{\infty} \mathfrak{g}_i; \quad \widetilde{\mathfrak{g}} = \prod_{i=-2}^{\infty} \widetilde{\mathfrak{g}}_i.$$

We will use the notation

$$(3.1.6) \quad \mathfrak{g}_{\geq 0} = \prod_{i=0}^{\infty} \mathfrak{g}_i; \quad \widetilde{\mathfrak{g}}_{\geq 0} = \prod_{i=0}^{\infty} \widetilde{\mathfrak{g}}_i.$$

Note that

$$(3.1.7) \quad \mathfrak{g}_0 \xrightarrow{\sim} \mathfrak{sp}(2n)$$

and the action of this Lie algebra on  $\widehat{\mathbb{A}}$  is the standard action of  $\mathfrak{sp}$  by infinitesimal linear coordinate changes.

**3.1.2. DG model for  $\mathbb{R}\text{Hom}(\mathbb{V}_0, \mathbb{V}_f)$ .** Though this is not needed for the sequel, let us explain how modules  $\mathbb{V}_f$  can be replaced by their DG analogs. Define

$$(3.1.8) \quad \Omega^\bullet(\mathbb{R}^{2n}, \widehat{\mathbb{V}}_f) = \Omega^\bullet(\mathbb{R}^n)[[\widehat{x}, \hbar]]$$

with the differential

$$(3.1.9) \quad \nabla_{\mathbb{V}} = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial \widehat{x}} \right) dx$$

and the action of  $\Omega^\bullet(\mathbb{R}^{2n}, \widehat{\mathbb{A}})$  defined as follows:  $x$  and  $\widehat{x}$  act by multiplication;  $\xi$  acts by multiplication by  $f'(x)$ ;  $\widehat{\xi}$  acts by  $i\hbar \frac{\partial}{\partial \widehat{x}} + f'(x + \widehat{x}) - f'(x)$ ;  $d\xi$  acts by  $df'(x) = f''(x)dx$ .

It is easy to see that  $\Omega^\bullet(\mathbb{R}^{2n}, \widehat{\mathbb{A}})$  is the space of global sections of a sheaf of differential graded algebras, and  $\Omega^\bullet(\mathbb{R}^{2n}, \widehat{\mathbb{V}}_f)$  is the space of global sections of a sheaf of differential graded modules supported on the Lagrangian submanifold  $L_f = \{\xi = f'(x)\}$ . The formula  $v \mapsto v(x + \widehat{x})$  defines a quasi-isomorphic embedding

$$\mathbb{V}_f \rightarrow \Omega(\mathbb{R}^n, \widehat{\mathbb{V}}_f)$$

compatible with the embedding of algebras  $C^\infty(\mathbb{R}^{2n})[[\hbar]] \rightarrow \Omega^\bullet(\mathbb{R}^{2n}, \widehat{\mathbb{A}})$  defined in (3.1.2).

**Lemma 3.1.** *Let  $e^*$ ,  $\widehat{e}^*$  and  $a^*$  be three free graded commutative variables of degrees 1, 1, and 0 respectively. The cohomology*

$$\mathbb{R}\mathrm{Hom}_{\Omega^\bullet(\mathbb{R}^{2n}, \widehat{\mathbb{A}})}(\Omega^\bullet(\mathbb{R}^{2n}, \widehat{\mathbb{V}}_0), \Omega^\bullet(\mathbb{R}^{2n}, \widehat{\mathbb{V}}_f))$$

*is computed by the complex*

$$\Omega^\bullet(\mathbb{R}^n, \widehat{\mathbb{V}}_f)[e^*, \widehat{e}^*][[a^*]], \nabla_{\mathbb{V}} + e^*\xi + \widehat{e}^*\widehat{\xi} + a^*d\xi + (e^* - \widehat{e}^*)\frac{\partial}{\partial a^*}$$

*which is isomorphic to  $\Omega^\bullet(\mathbb{R}^n)[[\widehat{x}, \hbar]][e^*, \widehat{e}^*][[a^*]]$  with the differential*

$$(\frac{\partial}{\partial x} - \frac{\partial}{\partial \widehat{x}})dx + e^*f'(x) + a^*f''(x)dx + \widehat{e}^*(i\hbar\frac{\partial}{\partial \widehat{x}} + f'(x + \widehat{x}) - f'(x)) + (e^* - \widehat{e}^*)\frac{\partial}{\partial a^*}$$

*The latter complex is quasi-isomorphic to the one in Lemma 2.2.*

*Proof.* The DG module  $\Omega^\bullet(\mathbb{R}^n, \widehat{\mathbb{V}}_0)$  is the quotient of the free DG module  $\Omega^\bullet(\mathbb{R}^{2n}, \widehat{\mathbb{A}})$  by the differential graded submodule generated by  $\xi$ ,  $d\xi$ , and  $\widehat{\xi}$ . A Koszul complex  $\mathcal{P} = \Omega^\bullet(\mathbb{R}^{2n}, \widehat{\mathbb{A}})[e, \widehat{e}, a]$  is a semi-free resolution of this quotient. The differential extends  $\nabla_{\mathbb{A}}$ , sends  $ev$  to  $\xi v + av$ ,  $\widehat{e}v$  to  $-\widehat{\xi}v + av$ ,  $av$  to  $d\xi \cdot v$ , and is a coderivation with respect to the action of  $\mathbb{C}[e, \widehat{e}, a]$ . The complex  $\mathrm{Hom}_{\Omega^\bullet(\mathbb{R}^{2n}, \widehat{\mathbb{A}})}(\mathcal{R}, \Omega^\bullet(\mathbb{R}^n, \widehat{\mathbb{V}}_f))$  is isomorphic to both complexes above. It remains to show that the latter of those complexes is quasi-isomorphic to  $(\Omega^\bullet(\mathbb{R}^{2n})[[\hbar]], i\hbar d_{\mathrm{DR}} + df \wedge)$ . To this end, consider the second complex in the statement of the lemma. Change the odd variables to  $e^*$  and  $e^* - \widehat{e}^*$ ; note that we can factor out all positive powers of  $a^*$  and  $e^* - \widehat{e}^*$ . This is because the differential  $(e^* - \widehat{e}^*)\frac{\partial}{\partial a^*}$  is acyclic. We are left with the complex  $\Omega^\bullet(\mathbb{R}^n)[[\widehat{x}, \hbar]][e^*]$  with differential

$$(\frac{\partial}{\partial x} - \frac{\partial}{\partial \widehat{x}})dx + e^*(i\hbar\frac{\partial}{\partial \widehat{x}} + f'(x + \widehat{x}))$$

Now change the even variables. Put  $y = x + \widehat{x}$  and keep  $\widehat{x}$  as the second variable. As for the odd variables, put  $Dx = dx - i\hbar e^*$  and keep  $e^*$  as the second variable. The differential becomes

$$(i\hbar\frac{\partial}{\partial y} + f'(y))e^* - \frac{\partial}{\partial \widehat{x}}Dx.$$

We can factor out all positive powers of  $\widehat{x}$  and of  $Dx$  because the differential  $\frac{\partial}{\partial \widehat{x}}Dx$  is acyclic.  $\square$

**3.2. Deformation quantization of symplectic manifolds.** We recall from [1] that a deformation quantization of a symplectic manifold  $M$  is a formal product

$$f * g = fg + \sum_{k=1}^{\infty} (i\hbar)^k P_k(f, g)$$

where  $P : C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)$  are bilinear bidifferential operators,  $f * (g * h) = (f * g) * h$  in  $C^\infty(M)[[\hbar]]$ ,  $1 * f = f * 1 = f$ , and

$$P_1(f, g) - P_1(g, f) = \{f, g\}.$$

An isomorphism between two deformation quantizations is a formal series

$$T(f) = f + \sum_{k=1}^{\infty} (i\hbar)^k T_k(f)$$

where  $T(f) * T(g) = T(f *' g)$  and  $T_k : C^\infty(M) \rightarrow C^\infty(M)$  are linear differential operators. Below we review how to classify deformation quantizations up to isomorphism using Fedosov connections.

**3.3. The bundle  $\widehat{\mathbb{A}}_M$ .** By  $\widehat{\mathbb{A}}_M$  we denote the bundle of algebras associated to the action of  $\mathrm{Sp}(2n)$  on  $\widehat{\mathbb{A}}$ .

**3.4. The Fedosov connection.**

**Definition 3.2.** A Fedosov connection  $\nabla$  is a connection in the bundle of algebras  $\mathcal{A}_M$  satisfying the following properties.

(1)

$$\nabla(fg) = \nabla(f)g + f\nabla(g)$$

for any local sections  $f$  and  $g$  of  $\mathbb{A}_M$ .

(2)  $\nabla^2 = 0$

(3) In any local Darboux coordinates  $x, \xi$  on  $M$  and any formal Darboux coordinates  $\widehat{x}, \widehat{\xi}$  of  $\mathbb{A}$ ,

$$\nabla = d_{\mathrm{DR}} - \left( \frac{\partial}{\partial \widehat{x}} dx - \frac{\partial}{\partial \widehat{\xi}} d\xi \right) + A_{\geq 0}$$

where  $A_{\geq 0}$  is a one-form with coefficients in  $\mathfrak{g}_{\geq 0}$  (we use the notation of (3.1.6)).

Note that  $\mathfrak{sp}(2n)$  embeds into  $\widetilde{\mathfrak{g}}$  as the space of  $\frac{1}{i\hbar}q(\widehat{x}, \widehat{\xi})$  where  $q$  is a quadratic polynomial.

**Definition 3.3.** A lifted Fedosov connection  $\widetilde{\nabla}$  is a collection of  $\widetilde{\mathfrak{g}}$ -valued one-forms  $A_j$  on local Darboux charts  $U_j$  such that

(1)

$$A_j = -dg_{jk}g_{jk}^{-1} + \mathrm{Ad}(g_{jk})A_k$$

for any  $j$  and  $k$ .

(2)  $\widetilde{\nabla}^2$  is central.

(3) In any local Darboux coordinates  $x, \xi$  on  $M$  and any formal Darboux coordinates  $\widehat{x}, \widehat{\xi}$  of  $\mathbb{A}$ ,

$$\widetilde{\nabla} = d_{\mathrm{DR}} - \frac{1}{i\hbar}\widehat{\xi}dx + \frac{1}{i\hbar}\widehat{x}d\xi + A_{\geq 0}$$

where  $A_{\geq 0}$  is a one-form with coefficients in  $\widetilde{\mathfrak{g}}_{\geq 0}$  (we use the notation of (3.1.6)).

Any lifted Fedosov connection  $\tilde{\nabla}$  defines a Fedosov connection  $\nabla$  via the projection  $\tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$ . In this case we call  $\tilde{\nabla}$  a lifting of  $\nabla$ .

Let

$$(3.4.1) \quad G = \mathrm{Sp}(2n, \mathbb{R}) \ltimes \exp(\mathfrak{g}_{\geq 1})$$

This group acts on  $\mathbb{A}$  by automorphisms. Let  $G_M$  be the associated bundle of groups. It acts by automorphisms on the bundle of algebras  $\mathbb{A}_M$ .

**Definition 3.4.** Two Fedosov connections are gauge equivalent if they are conjugated by a section of  $G_M$ .

**Theorem 3.5.** 1) For every

$$\theta = \frac{1}{i\hbar}\omega + \sum_{j=0}^{\infty} (i\hbar)^j \theta_j$$

where  $\theta_j$  are closed two-forms on  $M$ , there exists a lifted Fedosov connection  $\tilde{\nabla}$  such that  $\tilde{\nabla}^2 = \theta$ .

2) Any Fedosov connection has a lifting. Two Fedosov connections are gauge equivalent if and only if the curvatures of their liftings are cohomologous as  $\frac{1}{i\hbar}\mathbb{C}[[\hbar]]$ -valued two-forms. In particular, any Fedosov connection is locally gauge equivalent to the standard one.

3) For any Fedosov connection, the kernel of  $\nabla : \Omega_M^0(\mathbb{A}_M) \rightarrow \Omega_M^1(\mathbb{A}_M)$  is isomorphic to  $C_M^\infty[[\hbar]]$  as a sheaf of algebras. Therefore any Fedosov connection defines a deformation quantization of  $M$ .

4) Any deformation quantization comes from some Fedosov connection. Two deformation quantizations are isomorphic if and only if the corresponding Fedosov connections are gauge equivalent.

This is mostly contained in [9]. The complete proof can be found in [37]. See also [2].

#### 4. THE EXTENDED FEDOSOV CONSTRUCTION

**4.1. The algebra  $\mathcal{A}$ .** First consider a larger completion of the Weyl algebra. Recall that the assignment

$$(4.1.1) \quad |\hat{x}_j| = |\hat{\xi}_j| = 1; |\hbar| = 2$$

turns  $\hat{\mathbb{A}}$  into a complete graded algebra

$$(4.1.2) \quad \hat{\mathbb{A}} = \prod_{k=0}^{\infty} \hat{\mathbb{A}}_k$$

Let  $\hat{\mathbb{A}}[\hbar^{-1}]_k$  be the space of elements of degree  $k$  in  $\hat{\mathbb{A}}[\hbar^{-1}]$ .

Now define

$$(4.1.3) \quad \hat{\hat{\mathbb{A}}} = \left\{ \sum_{k=-N}^{\infty} a_k | a_k \in \hat{\mathbb{A}}[\hbar^{-1}]_k \right\}$$



where  $N$  runs through all integers. The product is the usual Moyal-Weyl product.

Now let  $\mathrm{Sp}^4(2n)$  be the group defined in 12.3 (in the case  $N = 4$ ). This group acts on  $\widehat{\mathbb{A}}$  through  $\mathrm{Sp}(2n)$ . Consider the cross product  $\mathrm{Sp}^4(2n) \ltimes \widehat{\mathbb{A}}$ .

*Remark 4.1.* Here and everywhere by cross products we will mean their completed versions. In other words, elements of the cross product are infinite sums  $\sum g_k a_k$  where  $g_k \in \mathrm{Sp}^4$ ,  $a_k \in \mathbb{A}[\hbar^{-1}]$ , and  $|a_k| \rightarrow \infty$ .

**Definition 4.2.**

$$\mathcal{A} = \left\{ \sum_{k=0}^{\infty} a_k e^{\frac{1}{i\hbar} c_k} \mid a_k \in \mathrm{Sp}^4(2n) \ltimes \widehat{\mathbb{A}}; c_k \in \mathbb{R}; c_k \rightarrow \infty \right\}$$

Let  $\mathcal{A}_{\Lambda}$  be defined exactly as above, but with an extra condition  $c_k \geq 0$ . We will sometimes write  $\mathcal{A}_{\mathbb{K}}$  instead of  $\mathcal{A}$ .

Note that we view  $\mathrm{Sp}^4(2n)$  as a *discrete* group.

**4.1.1. The Novikov ring.** Define

$$(4.1.4) \quad \Lambda = \left\{ \sum_{k=0}^{\infty} a_k e^{\frac{1}{i\hbar} c_k} \mid a_k \in \mathbb{C}; c_k \in \mathbb{R}; c_k \geq 0; c_k \rightarrow \infty \right\}$$

$$(4.1.5) \quad \mathbb{K} = \left\{ \sum_{k=0}^{\infty} a_k e^{\frac{1}{i\hbar} c_k} \mid a_k \in \mathbb{C}; c_k \in \mathbb{R}; c_k \rightarrow \infty \right\}$$

Clearly,  $\mathcal{A}$  is an algebra over  $\mathbb{K}$ .

**4.2. The bundle  $\mathcal{A}_M$ .** Since the action of  $\mathrm{Sp}(2n)$  extends from  $\widehat{\mathbb{A}}$  to  $\mathcal{A}$ , we get the associated bundle of algebras  $\mathcal{A}_M$  on any symplectic manifold  $M$ .

**4.3. The extended Fedosov connection.** Note that the action of the Lie algebra  $\tilde{\mathfrak{g}}$  extends to an action of  $\widehat{\mathbb{A}}$  and therefore any Fedosov connection  $\nabla_{\mathbb{A}}$  extends canonically to a connection that we denote by  $\nabla_{\mathcal{A}}$ .

## 5. ACTION UP TO INNER AUTOMORPHISMS

### 5.1. Groups acting up to inner automorphisms.

**Definition 5.1.** Let  $\Gamma$  be a group and  $A$  an associative algebra. An action of  $\Gamma$  on  $A$  up to inner automorphisms is the following data.

- 1) Automorphisms  $T_g : A \xrightarrow{\sim} A$  for all  $g \in \Gamma$ .
- 2) Invertible elements  $c(g_1, g_2)$  of  $A$  for all  $g_1, g_2$  in  $\Gamma$  such that

$$(5.1.1) \quad T_{g_1} T_{g_2} = \mathrm{Ad}(c(g_1, g_2)) T_{g_1 g_2}$$

$$(5.1.2) \quad c(g_1, g_2) c(g_1 g_2, g_3) = T_{g_1} c(g_2, g_3) c(g_1, g_2 g_3)$$

An *equivalence* between  $(T, c)$  and  $(T', c')$  is a collection  $\{b(g) \in A^\times | g \in G\}$  such that

$$(5.1.3) \quad T'_g = \text{Ad}(b(g))T_g; \quad c'(g_1, g_2) = b(g_1)T_{g_1}(b_{g_2})c(g_1, g_2)b(g_1g_2)^{-1}$$

It  $\{b'(g)\}$  is an equivalence between  $(T, c)$  and  $(T', c')$  and  $\{b''(g)\}$  is an equivalence between  $(T', c')$  and  $(T'', c'')$ , then their *composition* is defined by  $b(g) = b''(g)b'(g)$  and is an equivalence between  $(T, c)$  and  $(T'', c'')$ .

## 5.2. Derivations of square zero up to inner derivations.

**Definition 5.2.** Let  $A$  be a graded algebra and let  $\Gamma$  be a group acting on  $A$  up to inner automorphisms. A *derivation of  $A$  of square zero up to inner derivations compatible with the action of  $\Gamma$*  is the following data.

- 1) A derivation  $D$  of  $A$  of degree one;
- 2) an element  $R$  of  $A$  of degree two;
- 3) elements  $\alpha(g)$  of  $A$  of degree one for every element  $g$  of  $\Gamma$ , such that

$$D^2 = \text{ad}(R); \quad DR = 0; \quad T_gDT_g^{-1} = D + \text{ad}(\alpha(g));$$

$$D\alpha(g) + \alpha(g)^2 = T_gR - R;$$

$$\alpha(g_1) + T_{g_1}\alpha(g_2) - \text{Ad}(c(g_1, g_2))\alpha(g_1g_2) + Dc(g_1, g_2) \cdot c(g_1, g_2)^{-1} = 0.$$

Now assume that we are given two sets of data:  $(T, c)$  with a compatible  $(D, \alpha, R)$ , and  $(T', c')$  with a compatible  $(D', \alpha', R')$ . An equivalence

$$(T, c), (D, \alpha, R) \xrightarrow{\sim} (T', c'), (D', \alpha', R')$$

between them is an equivalence  $\{b(g)\}$  between the actions and an element  $\beta$  of  $A$  of degree one such that

$$(5.2.1) \quad D' = D + \text{ad}(\beta);$$

$$(5.2.2) \quad \alpha'(g) = -Db(g) \cdot b(g)^{-1} + \text{Ad}_{b(g)}(\alpha(g) + T_g\beta);$$

$$(5.2.3) \quad R' = R + D\beta + \beta^2$$

For two equivalences

$$(b'(g), \beta') : (T, c), (D, \alpha, R) \xrightarrow{\sim} (T', c'), (D', \alpha', R')$$

and

$$(b''(g), \beta'') : (T', c'), (D', \alpha', R') \xrightarrow{\sim} (T'', c''), (D'', \alpha'', R''),$$

their *composition* is an equivalence

$$(b(g), \beta) : (T, c), (D, \alpha, R) \xrightarrow{\sim} (T'', c''), (D'', \alpha'', R'')$$

given by

$$(5.2.4) \quad b(g) = b''(g)b'(g); \quad \beta = \beta'' + \beta'.$$

*Remark 5.3.* A graded algebra with  $D$  and  $R$  as in 1) and 2) subject to the first two equations in 3) is called a *curved differential graded algebra*. In other words, this is an  $A_\infty$  algebra with the only nonzero operations being  $m_0, m_1, m_2$ . Furthermore,  $(T_g, \alpha(g))$  are *curved morphisms*, i.e.  $A_\infty$  morphisms with the only nonzero operations  $T_0, T_1$ .

**5.2.1. Lie algebras acting up to inner derivations.** The above is a partial case of the following definition (that is not used in the sequel).

**Definition 5.4.** Consider an action of a group  $\Gamma$  on an algebra  $A$  given by the data  $T_g, c(g_1, g_2)$ . Let  $\mathcal{L}$  be a Lie algebra. An action of  $\mathcal{L}$  on  $A$  up to inner derivations compatible with the action of  $\Gamma$  is the following data.

- 1) A linear map  $D : \mathcal{L} \rightarrow \text{Der}(A)$ ,  $X \mapsto D_X$ ;
- 2) linear maps  $\alpha : \mathcal{L} \rightarrow A$  for any  $g \in \Gamma$ ,  $X \mapsto \alpha_X(g)$ .
- 3) a bilinear skew symmetric map  $R : \mathcal{L} \times \mathcal{L} \rightarrow A$ , satisfying

$$\begin{aligned} [D_X, D_Y] &= D_{[X, Y]} + \text{ad } R(X, Y); \\ D_X(R(Y, Z)) + D_Y(R(Z, X)) + D_Z(R(X, Y)) &= \\ &= [D_X, D_{[Y, Z]}] + [D_Y, D_{[Z, X]}] + [D_Z, D_{[X, Y]}]; \\ T_g D_X T_g^{-1} &= D + \text{ad}(\alpha_X(g)); \end{aligned}$$

$$\begin{aligned} D_X \alpha_Y(g) - D_Y \alpha_X(g) + [\alpha(X, g), \alpha(Y, g)] - \alpha_{[X, Y]}(g) &= T_g R(X, Y) - R(X, Y); \\ \alpha_X(g_1) + T_{g_1} \alpha_X(g_2) - \text{Ad}(c(g_1, g_2)) \alpha_X(g_1 g_2) + D_X c(g_1, g_2) \cdot c(g_1, g_2)^{-1} &= 0. \end{aligned}$$

More generally, let  $A$  be a graded algebra and  $\mathcal{L}$  is a graded Lie algebra. The above definition makes sense with the following changes:  $c(g_1, g_2)$  are of degree zero;  $R$  and  $\alpha$  are homogeneous of degree zero; and signs are present in the formulas. Definition 5.2 describes a partial case when  $\mathcal{L}$  is a one-dimensional graded Lie algebra concentrated in degree one.

**5.3. Modules with compatible structures.** For an algebra  $A$  with an action  $(T_g, c(g_1, g_2))$  of a group  $G$  up to inner automorphisms and for an  $A$ -module  $V$ , a compatible action of  $G$  on  $V$  is a collection  $\{T_g : V \rightarrow V | g \in G\}$  of module automorphisms such that  $T_{g_1} T_{g_2} = c(g_1, g_2) T_{g_1 g_2}$ .

Given a graded algebra  $A$  and a graded module  $V$  as above, consider a derivation  $(D_A, \alpha, R)$  of square zero of  $A$  up to inner derivations compatible with the action of  $G$ . A compatible derivation of  $V$  is a derivation  $D_V : V^\bullet \rightarrow V^{\bullet+1}$  such that

$$(5.3.1) \quad D_V^2 = R; \quad D_V(av) = D_A(a)v + (-1)^{|a|} a D_V(v); \quad T_g D_V T_g^{-1} = D_V + \alpha(g)$$

for all homogeneous  $a$  in  $A$  and  $v$  in  $V$ .

A morphism  $V \rightarrow W$  is by definition an  $A$ -module morphism commuting  $F$  such  $D_W F = F D_V$ .

Given an equivalence

$$(\{b(g)\}, \beta) : (T, c), (D, \alpha, R) \xrightarrow{\sim} (T', c'), (D', \alpha', R')$$

and an action and derivation on an  $A$ -module  $V$  compatible with  $(T, c)$ , then

$$(5.3.2) \quad T'_g = b(g)T_g; \quad D'_V = D_V + \beta$$

define on  $V$  an action and a derivation compatible with  $(T', c')$ . This operation is compatible with compositions of equivalences.

**5.4. Quotient groups acting up to inner automorphisms.** Assume given a surjection of groups  $G \rightarrow \Gamma$  with kernel  $H$ . Assume that  $A$  is an associative algebra together with a  $G$ -equivariant morphism of groups  $i : H \rightarrow A^\times$ . Consider an action of  $G$  on  $A$  by automorphisms,  $g \mapsto \mathbf{T}_g$ . This is of course a partial case of 5.1 with  $c(g_1, g_2) = 1$ .

Choose a section of  $G \rightarrow \Gamma$  sending  $g \in \Gamma$  to  $\bar{g} \in G$ . Put

$$(5.4.1) \quad T_g = \mathbf{T}_{\bar{g}}; \quad c(g_1, g_2) = i(\bar{g}_1 \bar{g}_2 (\overline{g_1 g_2})^{-1})$$

Furthermore, let  $\mathbf{D}, \beta(g), \mathbf{R}$  be a derivation of square zero up to inner derivations compatible with the action of  $G$ . Assume that

$$\beta(h) = -\mathbf{D}(ih)(ih)^{-1}$$

for all  $h \in H$ . Put

$$(5.4.2) \quad D = \mathbf{D}; \quad \alpha(g) = \beta(\bar{g}); \quad R = \mathbf{R}$$

**Lemma 5.5.** 1) *Formulas (5.4.2) define a derivation of square zero up to inner derivations compatible with the action of  $\Gamma$  given by (5.4.1). Given two different sections  $s_1 : g \mapsto \bar{g}$  and  $s_2 : g \mapsto \tilde{g}$ , formulas*

$$b(g) = i(\tilde{g} \bar{g}^{-1}); \quad \beta = 0$$

*define an equivalence  $B(s_2, s_1)$  between corresponding derivations. One has*

$$B(s_3, s_2)B(s_2, s_1) = B(s_3, s_1)$$

2) *Assume  $(V, \mathbf{T}_g, \mathbf{D}_V)$  is an  $A$ -module with a compatible action of  $G$  and with a compatible derivation. Put  $D_V = \mathbf{D}_V$ ;  $T_g = \mathbf{T}_{\bar{g}}$ . Then  $(V, T_g, D_V)$  is an  $A$ -module with a compatible action of  $\Gamma$  and a compatible derivation.*

The proof is straightforward.

There is also an analog of the above Lemma for Lie algebra actions as in 5.2.1.

**5.5. The case of groupoids.** Now let  $G$  be a groupoid with the set of objects  $X$ . Let  $A = \{A_x | x \in X\}$  be a family of algebras. An action of  $G$  on  $A$  up to inner automorphisms is the data consisting of operators  $T_g : A_x \xrightarrow{\sim} A_y$  for all  $g \in G_{x,y}$  and of invertible elements  $c(g_1, g_2) \in A_x$  for all  $g_1 \in G_{x_1, x_2}$  and  $g_2 \in G_{x_2, x_3}$  such that (5.1.2) is true. We give the same definition for a family  $A$  of graded algebras where we require  $c(g_1, g_2)$  to be of degree zero.

If  $A = \{A_x\}$  is a family of graded algebras with an action of  $G$  up to inner derivations, a derivation of square zero up to inner derivations compatible

with the action of  $A$  is a family of derivations  $\{D_x : A_x \rightarrow A_x | x \in X\}$  and of elements  $\{\alpha(g) \in A_{x_1} | x_1, x_2 \in X, g \in G_{x_1, x_2}\}$  such that

$$D_x^2 = \text{ad}(R_x); D_x R_x = 0; T_g D_{x_2} T_g^{-1} = D_{x_1} + \text{ad}(\alpha(g));$$

$$D\alpha(g) + \alpha(g)^2 = T_g R_{x_2} - R_{x_1};$$

$$\alpha(g_1) + T_{g_1} \alpha(g_2) - \text{Ad}(c(g_1, g_2)) \alpha(g_1 g_2) + D_{x_1} c(g_1, g_2) \cdot c(g_1, g_2)^{-1} = 0.$$

A similar definition can be given for a family of (graded) Lie algebras  $\{\mathcal{L}_x | x \in X\}$ .

Now consider a family of subgroups  $\{H_x \in G_{x,x} | x \in X\}$ , a groupoid  $\Gamma$  with the same set of objects  $X$ , and an epimorphism of groupoids  $G \rightarrow \Gamma$  such that  $H_x = \text{Ker}(G_{x,x} \rightarrow \Gamma_{x,x})$ . Let  $\{i_x : H_x \rightarrow A_x^\times\}$  be a  $G$ -equivariant family of morphisms of groups. Choose a section  $g \mapsto \bar{g}$  of  $G \rightarrow \Gamma$ .

**Lemma 5.6.** 1) Given an action  $\{\mathbf{T}_g\}$  of  $G$  on  $A$  with  $c(g_1, g_2) = 1$ , formulas (5.4.1) define an action of  $\Gamma$  on  $A$  up to inner automorphisms.

2) Given a derivation of square zero  $(\mathbf{D}, \mathbf{R}, \beta)$  up to inner derivations compatible with the action of  $G$ , assume that  $\beta(h) = -\mathbf{D}i(h) \cdot i(h)^{-1}$  for all  $x$  and all  $h \in H_x$ . Then formulas (5.4.2) define a derivation of square zero up to inner derivations compatible with the action of  $\Gamma$ .

3) For two different choices of sections  $s_1, s_2$ , same formulas as in Lemma 5.5, 1), define an equivalence  $B(s_2, s_1)$  between to derivations corresponding to two sections  $s_1, s_2$ . One has

$$B(s_3, s_2)B(s_2, s_1) = B(s_3, s_1).$$

**5.6. Modules with a compatible structure.** For  $A$  and  $G$  as in 5.5, an  $A$ -module  $V$  with a compatible action of  $G$  is a collection  $\{V_x | x \in X\}$  of  $A_x$  modules together with isomorphisms  $\{T_g : V_x \xleftarrow{\sim} V_y | x, y \in X; g \in G_{x,y}\}$  satisfying

$$T_g(av) = T_g(a)T_g(v); T_{g_1}T_{g_2} = c(g_1, g_2)T_{g_1 g_2}$$

If  $A$  and  $V$  are graded and  $(D_A, \alpha(g), R)$  is a compatible derivation of square zero up to inner derivations, a compatible derivation of  $V$  is a linear map  $D_V : V^\bullet \rightarrow V^{\bullet+1}$  such that

$$D_V^2 = R; D_V(av) = D_A(a)v + (-1)^{|a|}aD_V(v); T_g D_V T_g^{-1} = D_V + \alpha(g)$$

for all homogeneous  $a \in A_x, v \in V_x$ .

There are analogs of Lemma 5.5 that we leave to the reader.

## 5.7. The case of Lie groupoids.

**5.7.1. Lie groupoids: notation and conventions.** Recall that a groupoid with a set of morphisms  $\mathcal{G}$  and the set of objects  $M$  is a *Lie groupoid* [33] if  $\mathcal{G}$  and  $M$  are manifolds and the source and target maps  $s, t : \mathcal{G} \rightarrow M$  are smooth surjective submersions, and the composition, inverse, and the map  $M \rightarrow \mathcal{G}, x \mapsto \text{Id}_x$ , are smooth.

For two points  $x_0$  and  $x_1$  of  $M$ ,  $\mathcal{G}_{x_0, x_1} = \{g \in \mathcal{G} | t(g) = x_0, s(g) = x_1\}$ . This way, the composition is a map  $\mathcal{G}_{x_0, x_1} \times \mathcal{G}_{x_1, x_2} \rightarrow \mathcal{G}_{x_0, x_2}$ . If

$$\mathcal{G} \times_M \mathcal{G} = \{(g, g') \in \mathcal{G} \times \mathcal{G} | s(g) = t(g')\},$$

then the multiplication can be described as a map

$$m : \mathcal{G} \times_M \mathcal{G} \rightarrow \mathcal{G}.$$

We denote by  $\underline{\mathcal{G}}$  the sheaf of (pro)manifolds on  $M \times M$  defined by  $\underline{\mathcal{G}}(W) = (s, t)^{-1}(W)$ ,  $W \subset M \times M$ .

More generally, we have the map

$$\text{proj}_n : \mathcal{G} \times_M \dots \times_M \mathcal{G} \rightarrow M \times \dots \times M$$

where the product is  $n$ -fold on the left and  $(n+1)$ -fold on the right. In particular,  $\text{proj}_1 = (s, t)$ . Put

$$(5.7.1) \quad \underline{\mathcal{G}}^{(n)}(W) = \text{proj}_n^{-1}(W)$$

This is a sheaf of pro-manifolds on  $M^{n+1}$ .

By  $\mathcal{O}_M$  we denote a sheaf of (graded) algebras on  $M$  that could be  $C_M^\infty$ ,  $\Omega_M^\bullet$ , or the sheaf of  $\Lambda$ -valued forms or functions that we will consider later. All that we need is that  $\mathcal{O}_M$  be defined for every manifold  $M$  (of given type) and that for every morphism  $f : M \rightarrow N$  the inverse image  $f^*\mathcal{O}_N$  be defined, together with the morphisms  $f^{-1}\mathcal{O}_M \rightarrow f^*\mathcal{O}_M$  and  $f^*\mathcal{O}_N \rightarrow \mathcal{O}_M$  subject to the usual identities.

By  $p_j : M^{n+1} \rightarrow M$  we denote the projection onto the  $j$ th factor. Let  $\mathcal{A}$  be a sheaf of  $\mathcal{O}_M$ -algebras.

**Definition 5.7.** An action of  $\mathcal{G}$  on  $\mathcal{A}$  up to inner derivations is a morphism of sheaves on  $M \times M$

$$\underline{\mathcal{G}} \times p_2^*\mathcal{A} \rightarrow p_1^*\mathcal{A}; (g, a) \mapsto T_g a$$

and a morphism of sheaves on  $M \times M \times M$

$$c : \underline{\mathcal{G}}^{(2)} \rightarrow p_1^*\mathcal{A}$$

subject to

$$T_{g_1} T_{g_2}(a) = \text{Ad } c(g_1, g_2) T_{g_1 g_2}(a)$$

in  $p_1^*\mathcal{A}$ , for any local section  $a$  of  $p_3^*\mathcal{A}$  and any two local sections  $g_2$  of  $p_{23}^*\underline{\mathcal{G}}$  and  $g_1$  of  $p_{12}^*\underline{\mathcal{G}}$ .

*Remark 5.8.* Given two local sections  $g_1, g_2$  as above, by their composition we mean the following. If  $g_1 = g_1(x_1, x_2, x_3) \in \mathcal{G}_{x_1, x_2}$  and  $g_2 = g_2(x_1, x_2, x_3) \in \mathcal{G}_{x_2, x_3}$ , then  $(g_1 g_2)(x_1, x_2, x_3) = g_1(x_1, x_2, x_3) g_2(x_1, x_2, x_3)$  in  $G_{x_1, x_3}$ . Similarly for  $c(g_1, g_2)$ .

**5.7.2. Flat connections up to inner derivations.** Here we assume that  $\mathcal{O}_M$ , or  $\mathcal{O}_M^\bullet$ , is a differential graded algebra with a differential  $d$ . A *connection* on a sheaf of graded  $\mathcal{O}_M^\bullet$ -modules  $\mathcal{E}$  is a morphism of sheaves  $\nabla : \mathcal{E} \rightarrow \mathcal{E}$  of degree one such that  $\nabla(ae) = da \cdot e + (-1)^{|a|} a \nabla e$ .

We also assume that for every  $f : M \rightarrow N$  and every sheaf of graded  $\mathcal{O}_N^\bullet$ -modules  $\mathcal{E}$ , a natural connection  $f^*\nabla$  on  $f^*\mathcal{E}$  is defined, subject to the usual properties. For us  $\mathcal{O}_M^\bullet$  will be the sheaf of  $\Lambda$ -valued forms, and  $f^*\nabla$  will be a straightforward analog of the standard inverse image of a connection that we will define in 8.1.

**Definition 5.9.** Let  $\mathcal{A}^\bullet$  is a sheaf of graded  $\mathcal{O}_M^\bullet$ -algebras with an action of  $\mathcal{G}$  up to inner automorphisms. A flat connection up to inner derivations compatible with the action of  $\mathcal{G}$  is the following data.

- 1) A connection  $\nabla : \mathcal{A}^\bullet \rightarrow \mathcal{A}^{\bullet+1}$  which is a derivation.
- 2) A section  $R$  of  $\mathcal{A}^2$ .
- 3) A morphism of sheaves  $\alpha : \underline{\mathcal{G}} \rightarrow p_1^*\mathcal{A}^\bullet$  of degree one, such that:

$$\begin{aligned} \nabla^2 &= \text{ad}(R); \quad \nabla R = 0; \quad T_g(p_2^*\nabla)T_g^{-1} = p_1^*\nabla + \text{ad}(\alpha(g)); \\ (p_1^*\nabla)\alpha(g) + \alpha(g)^2 &= T_g(p_2^*R) - p_1^*R; \end{aligned}$$

$$\alpha(g_1) + T_{g_1}\alpha(g_2) - \text{Ad}(c(g_1, g_2))\alpha(g_1g_2) + (p_1^*\nabla)c(g_1, g_2) \cdot c(g_1, g_2)^{-1} = 0.$$

We will often write  $\alpha(g) = -\nabla g \cdot g^{-1}$ .

### 5.8. Modules with a compatible structure: the Lie groupoid case.

In the situation of Definition 5.9, let  $(\mathcal{V}^\bullet, \nabla_{\mathcal{V}})$  be a differential graded  $\mathcal{A}^\bullet$ -module together with a morphism of sheaves  $M \times M$

$$\underline{\mathcal{G}} \times p_2^*\mathcal{V} \rightarrow p_1^*\mathcal{V}; \quad (g, v) \mapsto T_g v$$

subject to:

$$T_{g_1}T_{g_2}(v) = c(g_1, g_2)T_{g_1g_2}(v)$$

in  $p_1^*\mathcal{V}^\bullet$ , for any local section  $v$  of  $p_3^*\mathcal{V}$  and any two local sections  $g_2$  of  $p_{23}^*\underline{\mathcal{G}}$  and  $g_1$  of  $p_{12}^*\underline{\mathcal{G}}$ ;

$$T_g(av) = T_g(a)T_g(v)$$

in  $p_1^*\mathcal{V}$ , for any local sections  $a$  of  $p_2^*\mathcal{A}^\bullet$  and  $v$  of  $p_2^*\mathcal{V}^\bullet$ ;

$$\nabla_{\mathcal{V}}^2 = R; \quad \nabla_{\mathcal{V}}(av) = \nabla_{\mathcal{A}}(a)v + (-1)^{|a|} a \nabla_{\mathcal{V}}(v)$$

for any homogeneous local sections  $a$  of  $\mathcal{A}^\bullet$  and  $v$  of  $\mathcal{V}^\bullet$ ;

$$T_g(p_2^*D_V)T_g^{-1} = \pi_1^*D_V + \alpha(g)$$

**5.8.1. The action of the quotient in the Lie groupoid case.** Now consider two Lie groupoids  $\mathcal{G}$  and  $\Gamma$  with the same manifold of objects  $M$  and an epimorphism of groupoids  $\mathcal{G} \rightarrow \Gamma$  (over  $M$ .) Define  $\mathcal{H}_x = \text{Ker}(\mathcal{G}_{x,x} \rightarrow \Gamma_{x,x})$  and  $\mathcal{H} = \cup_{x \in M} \mathcal{H}_x$ . Consider the morphism  $\mathcal{H} \rightarrow M$ . Define the sheaf of groups  $\underline{\mathcal{H}}(U) = s^{-1}(U)$  for  $U \subset M$ . Let  $i : \underline{\mathcal{H}} \rightarrow \mathcal{A}^\times$  be a  $\mathcal{G}$ -equivariant morphism of sheaves of groups. Choose a section  $g \mapsto \bar{g}$  of  $\mathcal{G} \rightarrow \Gamma$ .

**Lemma 5.10.** 1) Given an action  $\{\mathbf{T}_g\}$  of  $\mathcal{G}$  on  $\mathcal{A}$  with  $c(g_1, g_2) = 1$ , formulas (5.4.1) define an action of  $\Gamma$  on  $\mathcal{A}$  up to inner automorphisms.

2) Given a flat connection  $(\mathbf{D}, \mathbf{R}, \beta)$  up to inner derivations compatible with the action of  $\mathcal{G}$ , assume that  $\beta(h) = -\mathbf{D}i(h) \cdot i(h)^{-1}$  for all local sections of  $\mathcal{H}$ . Then formulas

$$\nabla = \mathbf{D}; \alpha(g) = \beta(\bar{g}); R = \mathbf{R}$$

define a flat connection up to inner derivations compatible with the action of  $\Gamma$ .

3) For two different choices of sections  $s_1, s_2$ , same formulas as in Lemma 5.5, 1), define an equivalence  $B(s_2, s_1)$  between to derivations corresponding to two sections  $s_1, s_2$ . One has

$$B(s_3, s_2)B(s_2, s_1) = B(s_3, s_1).$$

4) Let  $\mathcal{V}$  be a graded  $\mathcal{A}$ -module with a compatible action  $\mathbf{T}$  of  $\mathcal{G}$  and a compatible connection  $\mathbf{D}_{\mathcal{V}}$ . Then formulas

$$T_g = \mathbf{T}_{\bar{g}}; \nabla_{\mathcal{V}} = \mathbf{D}_{\mathcal{V}}$$

define a compatible action of  $\Gamma$  and a compatible connection on  $\mathcal{V}$ .

*Remark 5.11.* Note that the morphisms of sheaves  $c : \underline{\Gamma}^{(2)} \rightarrow p_1^* \mathcal{A}$  and  $\alpha : \underline{\Gamma} \rightarrow p_1^* \mathcal{A}$  are discontinuous. For us  $\Gamma$  will be an étale groupoid, more precisely the fundamental groupoid of  $M$ . We can only make a choice of a continuous  $c$  and  $\alpha$  on any small coordinate chart, but that will be enough for our purposes. More precisely, this will define to a *twisted*  $A_{\infty}$  action as it is explained in 16.

## 6. FROM ACTIONS UP TO INNER AUTOMORPHISMS TO $A_{\infty}$ ACTIONS

It is a well-known fact that inner isomorphisms act on the Ext functors trivially. Therefore, if a group acts on an algebra up to inner automorphisms, given compatible actions on two  $A$ -modules  $V$  and  $W$ , the group acts on the cohomology  $\text{Ext}_A^{\bullet}(V, W)$ . In this section we prove a more precise version of this fact, namely we construct an  $A_{\infty}$  action of the group on the standard bar complex.

**6.1.  $A_{\infty}$  actions.** An  $A_{\infty}$  action of a group  $G$  on a complex  $C^{\bullet}$  is a collection  $\{T(g_1, \dots, g_n) \in \text{Hom}^{1-n}(C^{\bullet}, C^{\bullet}) | g_1, \dots, g_n \in G, n > 0\}$  satisfying

$$(6.1.1) \quad [d, T(g_1, \dots, g_n)] + \sum_{j=1}^{n-1} (-1)^j T(g_1, \dots, g_j) T(g_{j+1}, \dots, g_n) -$$

$$\sum_{j=1}^{n-1} (-1)^j T(g_1, \dots, g_j g_{j+1}, \dots, g_n) = 0$$

We sometimes write  $T_g$  instead of  $T(g)$ . The operators  $T(g)$  induce an action of  $G$  on the cohomology of  $C^{\bullet}$ .



An  $A_\infty$  morphism between two  $A_\infty$  actions  $T$  and  $T'$  is a collection  $\{\phi(g_1, \dots, g_n) \in \text{Hom}^{-n}(C^\bullet, C^\bullet) | g_1, \dots, g_n \in G, n \geq 0\}$  satisfying

$$(6.1.2) \quad [d, \phi(g_1, \dots, g_n)] + \sum_{j=1}^{n-1} (-1)^j T'(g_1, \dots, g_j) \phi(g_{j+1}, \dots, g_n) - \\ - \sum_{j=1}^{n-1} (-1)^j \phi(g_1, \dots, g_j) T(g_{j+1}, \dots, g_n) - \sum_{j=1}^{n-1} (-1)^j \phi(g_1, \dots, g_j g_{j+1}, \dots, g_n) = 0$$

**6.2. The Ext functors.** Let  $A$  be an associative algebra and  $V, W$  two  $A$ -modules. By  $C^\bullet(V, A, W)$ , or simply  $C^\bullet(V, W)$ , we denote the standard complex computing  $\text{Ext}_A^\bullet(V, W)$ . Namely,

$$C^m(V, W) = \prod_{p+n=m} \text{Hom}(A^{\otimes n}, \text{Hom}^p(V, W));$$

the differential  $\delta$  is defined by

$$(\delta\varphi)(a_1, \dots, a_{n+1}) = (-1)^{|\varphi||a_1|} a_1 \varphi(a_2, \dots, a_{n+1}) + \\ \sum_{j=1}^n (-1)^{\sum_{i=1}^j (|a_i|+1)} \varphi(a_1, \dots, a_j a_{j+1}, \dots, a_{n+1}) + \\ (-1)^{\sum_{i=1}^{n+1} (|a_i|+1)} \varphi(a_1, \dots, a_n) a_{n+1}$$

**Lemma 6.1.** 1) Let  $T$  be an automorphism of  $A$  together with compatible automorphisms of  $V$  and  $W$  (i.e. invertible operators  $T$  such that  $T(av) = T(a)T(v)$ ). Put

$$(T\varphi)(a_1, \dots, a_n) = T\varphi(T^{-1}a_1, \dots, T^{-1}a_n)T^{-1}$$

Then  $\varphi \mapsto T\varphi$  is an automorphism of  $C^\bullet(V, W)$ .

2) For an invertible element  $c$  of  $A$  of degree zero define

$$(\phi(c)\varphi)(a_1, \dots, a_n) = \\ = - \sum_{j=0}^n (-1)^{\sum_{i=1}^j (|a_i|+1)} \varphi(a_1, \dots, a_j, c, c^{-1}a_{j+1}c, \dots, c^{-1}a_nc)c^{-1}$$

One has

$$[\delta, \phi(c)] = \text{Ad}(c) - \text{Id}$$

3) More generally, for  $m$  invertible elements  $c_1, \dots, c_m$  of degree zero of  $A$ , define

$$(\phi(c_1, \dots, c_m)\varphi)(a_1, \dots, a_n) = \\ = - \sum_{0 \leq j_1 \leq \dots \leq j_m \leq n} (-1)^{\sum_{k=1}^m \sum_{i=1}^{j_k} (|a_i|+1)} \varphi(a_1, \dots, a_{j_1}, c_1, c_1^{-1}a_{j_1+1}c_1, \dots, c_1^{-1}a_{j_2}c_1, \\ c_2, (c_1c_2)^{-1}a_{j_2+1}(c_1c_2), \dots, (c_1c_2)^{-1}a_{j_3}(c_1c_2), \dots, \\ c_m, (c_1 \dots c_m)^{-1}a_{j_m+1}(c_1 \dots c_m), \dots, (c_1 \dots c_m)^{-1}a_n(c_1 \dots c_m))(c_1 \dots c_m)^{-1}$$

One has

$$[d, \phi(c_1, \dots, c_m)] + \text{Ad}_{c_1} \phi(c_2, \dots, c_m) + \sum_{j=1}^{m-1} (-1)^j \phi(c_1, \dots, c_j c_{j+1}, \dots, c_m) + (-1)^m \phi(c_1, \dots, c_{m-1}) = 0$$

In other words: the group of automorphisms of  $(A, V, W)$  acts on  $C^\bullet(V, A, W)$ ; the subgroup of inner automorphisms acts homotopically trivially, in the sense that there is an  $A_\infty$  morphism, starting with the identity, between this action and the trivial action. Note that, as in 1) above, we denote by  $\text{Ad}_c$  both the inner automorphism of  $A$  and the induced automorphism of  $C^\bullet(V, A, W)$ .

**Lemma 6.2.**

$$\phi(c_1, \dots, c_m) \phi(d_1, \dots, d_n) = \sum \pm \phi(e_1, \dots, e_{n+m})$$

where the summation is over all  $(e_1, \dots, e_{n+m})$  such that:

- a) as a set,  $\{e_1, \dots, e_{n+m}\} = \{d_1, \dots, d_m, x_1 c_1 x_1^{-1}, \dots, x_n c_n x_n^{-1}\}$ , with  $x_j$  defined below in c);
- b) the order of elements  $d_j$  is preserved; the order of the elements  $x_j c_j x_j^{-1}$  is the same as the order of the elements  $c_j$ ;
- c)  $x_j$  is the product of all  $d_k$  to the left of  $x_j c_j x_j^{-1}$ . The sign is  $(-1)^N$  where  $N$  is the number of instances when  $d_k$  is to the left of  $x_j c_j x_j^{-1}$ .

For example,

$$\phi(c) \phi(d) = \phi(c, d) - \phi(d, d c d^{-1})$$

### 6.2.1. A lemma about $A_\infty$ actions.

**Lemma 6.3.** Let  $\tilde{G}$  be a group and  $H$  its normal subgroup. Let  $G = \tilde{G}/H$ . Consider a complex  $C^\bullet$  with the following data:

- 1) An action of  $\tilde{G}$ ,  $g \mapsto \mathcal{T}_g$  for any  $g \in \tilde{G}$ .
- 2) Operators  $\Phi(c_1, \dots, c_m) : C^\bullet \rightarrow C^{\bullet-m}$ ,  $m \geq 0$ , for all  $c_1, \dots, c_m \in H$ , satisfying

$$[d, \Phi(c_1, \dots, c_m)] + \mathcal{T}_{c_1} \Phi(c_2, \dots, c_m) + \sum_{j=1}^{m-1} (-1)^j \Phi(c_1, \dots, c_j c_{j+1}, \dots, c_m) + (-1)^m \Phi(c_1, \dots, c_{m-1}) = 0$$

$$\Phi(c_1, \dots, c_m) \Phi(d_1, \dots, d_n) = \sum \pm \Phi(e_1, \dots, e_{n+m})$$

as in Lemma 6.2. For any section  $g \mapsto \bar{g}$  of the projection  $\tilde{G} \rightarrow G$ , there is an  $A_\infty$  action of  $G$  on  $C^\bullet$  such that  $T_g = \mathcal{T}_{\bar{g}}$ .

*Proof.* Consider the differential graded algebra  $\mathcal{B}(H, \tilde{G})$  generated by the group algebra of  $\tilde{G}$  and by elements  $\Phi(c_1, \dots, c_m)$  of degree  $-m$  for all  $c_1, \dots, c_m$  in  $H$ , such that:

a)

$$g\Phi(c_1, \dots, c_m)g^{-1} = \Phi(gc_1g^{-1}, \dots, gc_mg^{-1})$$

for any  $g \in \tilde{G}$ ;b) the differential  $\partial$  satisfies

$$\begin{aligned} & \partial\Phi(c_1, \dots, c_m) + c_1\Phi(c_2, \dots, c_m) + \\ & + \sum_{j=1}^{m-1} (-1)^j \Phi(c_1, \dots, c_j c_{j+1}, \dots, c_m) + (-1)^m \Phi(c_1, \dots, c_{m-1}) = 0 \end{aligned}$$

c)

$$\Phi(c_1, \dots, c_m)\Phi(d_1, \dots, d_n) = \sum \pm \Phi(e_1, \dots, e_{n+m})$$

as in Lemma 6.2. This differential graded algebra is quasi-isomorphic to  $k[G]$ . In fact, as a complex it is the standard bar construction of  $H$  with coefficients in the right module  $k[\tilde{G}]$ . The quasi-isomorphism is the morphism of algebras such that

$$(6.2.1) \quad \Phi(c_1, \dots, c_m) \mapsto 0; g \mapsto \text{proj}_G(g), g \in \tilde{G}.$$

There is (unique up to homotopy) morphism from the standard resolution  $\text{CobarBar}(k[G])$  to  $\mathcal{B}(H, \tilde{G})$  over  $k[G]$ . Now define  $T(g_1, \dots, g_n)$  to be the action of the image of the generator  $(g_1 | \dots | g_n)$  on  $C^\bullet$ .  $\square$

**6.2.2. The  $A_\infty$  action on the standard complex.** Now assume that a group  $G$  acts on an algebra  $A$  up to inner automorphisms. Assume that  $V$  and  $W$  are two  $A$ -modules with compatible actions. This means that there are linear automorphisms  $T_g$  of  $V$  and  $W$  for any  $g \in G$  such that

$$(6.2.2) \quad T_g(av) = T_g(a)T_g(v); T_{g_1}T_{g_2} = c(g_1, g_2)T_{g_1g_2}$$

( $c(g_1, g_2)$  in the right hand side denotes the module action of the element of  $A$ ).

**Theorem 6.4.** *There is an  $A_\infty$  action of  $G$  on  $C^\bullet(V, A, W)$  such that  $T(g)$  is equal to  $T_g$  as in 6.1.*

*Proof.* Let  $\tilde{G} = G \ltimes_c A^\times$  be the group whose elements are expressions  $ag$ ,  $g \in G$  and  $A^\times$ , with the product

$$(6.2.3) \quad (a_1g_1)(a_2g_2) = a_1T_{g_1}(a_2)c(g_1, g_2)(g_1g_2)$$

and  $H = A^\times$ . The theorem follows immediately from Lemmas 6.1, 6.2, and 6.3.  $\square$

*Remark 6.5.* The proof of Theorem 6.4 actually leads to a rather simple recursive formula for the  $A_\infty$  action. Namely, the construction of a morphism

$$(6.2.4) \quad \text{CobarBar}(k[G]) \rightarrow \mathcal{B}(A^\times, G \ltimes_c A^\times)$$

(see the proof of Lemma 6.3) is an inductive procedure in  $n$  for finding the image of  $(g_1 | \dots | g_n)$  under this morphism. Let us describe this procedure. Consider the subalgebra  $\mathcal{B}(A^\times, A^\times)$  of expressions  $c_0 \Phi(c_1, \dots, c_m)$ . This subalgebra is quasi-isomorphic to  $k$ , the homotopy being

$$(6.2.5) \quad s(c_0 \Phi(c_1, \dots, c_m)) = \Phi(c_0, c_1, \dots, c_m)$$

Now define  $\Psi(g_1, \dots, g_n)$  in  $\mathcal{B}(A^\times, A^\times)$  recursively by

$$(6.2.6) \quad \Psi_1(g) = g;$$

$$(6.2.7) \quad \Psi(g_1, \dots, g_{n+1}) =$$

$$s \sum_{j=1}^n (-1)^j \Psi(g_1, \dots, g_j) T_{g_1 \dots g_j} \Psi(g_{j+1}, \dots, g_{n+1}) c(g_1 \dots g_j, g_{j+1} \dots g_{n+1})$$

Here the product is described in Lemma 6.2, and

$$T_g(c_0 \Phi(c_1, \dots, c_m)) = (T_g c_0 \Phi(T_g c_1, \dots, T_g c_m))$$

The elements  $\Psi(g_1, \dots, g_n)$  is some linear combinations of  $\phi(c_1, \dots, c_k)$  and  $c_j$  are algebraic expressions in  $T_{h_0} c(h_1, h_2)$ ,  $h_i$  being some products of  $g_i$ .

Let  $\psi(g_1, \dots, g_n)$  be the image of  $\Psi(g_1, \dots, g_n)$  under the morphism of algebras

$$(6.2.8) \quad \mathcal{B}(A^\times, A^\times) \rightarrow \text{End}(C^\bullet)$$

sending  $g \in G$  to  $T_g$ ,  $c \in A^\times$  to  $\text{Ad}(c)$ , and  $\Phi(g_1, \dots, g_n)$  to  $\phi(g_1, \dots, g_n)$ . Then

$$T(g_1, \dots, g_n) = \psi(g_1, \dots, g_n) T_{g_1 \dots g_n}$$

For example,

$$T(g_1, g_2) = \phi(c(g_1, g_2)) T_{g_1 g_2}$$

**6.2.3. The case of groupoids.** Let  $G$  be a groupoid with the set of objects  $X$  that acts on a family of algebras  $A = \{A_x | x \in X\}$  up to inner automorphisms. Let  $V = \{V_x | x \in X\}$  and  $W = \{W_x | x \in X\}$  two  $A$ -modules with compatible actions of  $G$ , *i.e.* with families  $T_g : V_x \xleftarrow{\sim} V_y$  and  $T_g : W_x \xleftarrow{\sim} W_y$ , satisfying (6.2.2).

Given a family of complexes  $C^\bullet = \{C_x^\bullet | x \in X\}$ , an  $A_\infty$  action of  $G$  on  $C^\bullet$  is a collection of

$$T(g_1, \dots, g_n) : C_{x_{n+1}}^{\bullet+1-n} \xleftarrow{\sim} C_{x_1}^\bullet$$

for any  $g_j \in G_{x_j, x_{j+1}}$ ,  $j = 1, \dots, n$ , satisfying the identities in the beginning of 6.1. Morphisms between  $A_\infty$  actions are defined similarly.

Define

$$C^\bullet(V, A, W)_x = C^\bullet(V_x, A_x, W_x)$$

**Theorem 6.6.** *There is an  $A_\infty$  action of  $G$  on  $C^\bullet(V, A, W)$  such that  $T(g)$  is equal to  $T_g$  as in 6.1.*

The proof is identical to the proof of Theorem 6.4.

**6.2.4.  $A_\infty$  action on the standard complex and derivations.** Let  $A$  be a graded algebra with an action of  $G$  up to inner automorphisms. Let  $D$  be a compatible derivation of square zero up to inner derivations. If  $V$  and  $W$  are two graded  $A$ -modules with compatible actions of  $G$ , we assume that they carry a compatible derivation, *i.e.* an operator  $D : V \rightarrow V$  or  $W \rightarrow W$  of degree one satisfying

$$(6.2.9) \quad D(av) = D(a)v + (-1)^{|a|}aD(v); \quad D^2 = R; \quad T_gDT_g^{-1} = D + \alpha(g)$$

Here  $R$  and  $\alpha(g)$  stand for the action of corresponding elements of  $A$ . For any homogeneous derivation  $E$  of  $A$  that acts on  $V$  and  $W$  compatibly, put

$$(6.2.10) \quad (E\varphi)(a_1, \dots, a_n) = [E, \varphi(a_1, \dots, a_n)] - \sum_{j=1}^n (-1)^{\sum_{i=1}^{j-1} |E|(|a_i|+1)} \varphi(a_1, \dots, Ea_j, \dots, a_n)$$

Put for any homogeneous element  $a$  of  $A$  put

$$(6.2.11) \quad (\iota_a \varphi)(a_1, \dots, a_n) = \sum_{j=0}^n (-1)^{\sum_{i=1}^j (|a|+1)(|a_i|+1)} \varphi(a_1, \dots, a_j, a, a_{j+1}, \dots, a_n)$$

**Lemma 6.7.**

$$[\delta, E] = 0; \quad [\delta, \iota_a] = \text{ad}(a); \quad [E, \iota_a] = (-1)^{|E|} \iota_{Da}; \quad [\iota_a, \iota_b] = 0.$$

**Corollary 6.8.**

$$(\delta + D - \iota_R)^2 = 0$$

on  $C^\bullet(V, A, W)$ .

*Remark 6.9.* We will always consider  $C^\bullet(V, A, W)$  as the standard complex equipped with the total differential  $\delta + D - \iota_R$ .

We now define an  $A_\infty$  action on this standard complex. We follow the proof of Theorem 6.4. The only change is a different choice of operators  $T_g$  and  $\phi(c_1, \dots, c_n)$  (see Lemma 6.1, 3)).

$$(6.2.12) \quad \mathcal{T}_g = \exp(\iota_{\alpha(g)}) T_g$$

for every  $g \in G$ ;

$$(6.2.13) \quad \widetilde{\text{Ad}}(c) = \exp(-\iota_{Dc \cdot c^{-1}}) \text{Ad}(c)$$

for every  $c \in A^\times$  of degree zero.

**Lemma 6.10.** a)  $[\delta, \iota_a] = \text{ad}_a$ ;  $[E, \iota_a] = (-1)^{|E|} \iota_{Da}$ ;  $[\iota_{a_1}, \iota_{a_2}] = 0$ ;

b)  $[\delta + D - \iota_R, \mathcal{T}_g] = 0$ ;

c)  $\widetilde{\text{Ad}}(c_1) \widetilde{\text{Ad}}(c_2) = \widetilde{\text{Ad}}(c_1 c_2)$ ;

d)  $\mathcal{T}_g \widetilde{\text{Ad}}_c \mathcal{T}_g^{-1} = \widetilde{\text{Ad}}_{T_g c} c T_{g_1} \mathcal{T}_{g_2} = \widetilde{\text{Ad}}(c(g_1, g_2)) \mathcal{T}_{g_1} \mathcal{T}_{g_2}$

*Proof.* a) is straightforward. Let us prove b).

$$\begin{aligned}\mathcal{T}_g(\delta + D - \iota_R)\mathcal{T}_g^{-1} &= e^{\iota_{\alpha(g)}}T_g(\delta + D - \iota_R)T_g^{-1}e^{-\iota_{\alpha(g)}} = \\ &= e^{\iota_{\alpha(g)}}(\delta + D + \mathbf{ad}_{\alpha(g)} - \iota_{R+D\alpha(g)+\alpha(g)^2})e^{-\iota_{\alpha(g)}}\end{aligned}$$

(we used the equations in Definition 5.2). Now observe that

$$e^{\iota_{\alpha(g)}}De^{-\iota_{\alpha(g)}} = D + \iota_{\alpha(g)};$$

$$e^{\iota_{\alpha(g)}}\delta e^{-\iota_{\alpha(g)}} = \delta - \mathbf{ad}_{\alpha(g)} + \iota_{\alpha(g)^2}$$

which implies b).

Now prove c).

$$\begin{aligned}\widetilde{\mathrm{Ad}}_{c_1}\widetilde{\mathrm{Ad}}_{c_2} &= \exp(-\iota_{Dc_1 \cdot c_1^{-1}})\mathrm{Ad}_{c_1}\exp(-\iota_{Dc_2 \cdot c_2^{-1}})\mathrm{Ad}_{c_2} = \\ &= \exp(-\iota_{Dc_1 \cdot c_1^{-1} + \mathrm{Ad}_{c_1}(Dc_2 \cdot c_2^{-1})})\mathrm{Ad}_{c_1c_2} = \\ &= \exp(-\iota_{D(c_1c_2) \cdot (c_1c_2)^{-1}})\mathrm{Ad}_{c_1c_2} = \widetilde{\mathrm{Ad}}_{c_1c_2}\end{aligned}$$

Next, observe that, because of the third equation in Definition 5.2,

$$\begin{aligned}T_g(Dc \cdot c^{-1}) &= T_g(Dc)T_g(c)^{-1} = D(T_g(c))T_g(c)^{-1} + [\alpha(g), T_g(c)]T_g(c)^{-1} = \\ &= D(T_g(c))T_g(c)^{-1} + \alpha(g) - \mathrm{Ad}_{T_g(c)}(\alpha(g))\end{aligned}$$

which implies

$$\begin{aligned}\mathcal{T}_g\widetilde{\mathrm{Ad}}_c\mathcal{T}_g^{-1} &= e^{\iota_{\alpha(g)}}T_g e^{-\iota_{Dc \cdot c^{-1}}}\mathrm{Ad}_c T_g^{-1}e^{-\iota_{\alpha(g)}} = \\ &= \exp(\iota_{\alpha(g)} - \iota_{T_g(Dc \cdot c^{-1})})\mathrm{Ad}_{T_g(c)}\exp(-\iota_{\alpha(g)}) = \\ &= \exp(\iota_{\alpha(g)} - \iota_{T_g(Dc \cdot c^{-1})} - \iota_{T_g(\alpha(g))})\mathrm{Ad}_{T_g(c)} = \\ &= \exp(-\iota_{DT_g(c) \cdot T_g(c)^{-1}})\mathrm{Ad}_{T_g(c)} = \widetilde{\mathrm{Ad}}_{T_g(c)}\end{aligned}$$

which is d). Finally,

$$\begin{aligned}\mathcal{T}_{g_1}\mathcal{T}_{g_2} &= \exp(\iota_{\alpha(g_1)})T_{g_1}\exp(\iota_{\alpha(g_2)})T_{g_2} = \exp(\iota_{\alpha(g_1)+T_{g_1}\alpha(g_2)})T_{g_1g_2} = \\ &+ \exp(\iota_{\alpha(g_1)+T_{g_1}\alpha(g_2)})\mathrm{Ad}_{c(g_1,g_2)}T_{g_1}T_{g_2}\end{aligned}$$

while

$$\begin{aligned}\mathrm{Ad}_{c(g_1,g_2)}\mathcal{T}_{g_1g_2} &= \exp(-\iota_{Dc(g_1,g_2)c(g_1,g_2)^{-1}})\mathrm{Ad}_{c(g_1,g_2)}\exp(\iota_{\alpha(g_1g_2)})T_{g_1g_2} = \\ &= \exp(-\iota_{Dc(g_1,g_2)c(g_1,g_2)^{-1}} - \iota_{\mathrm{Ad}_{c(g_1,g_2)}(\alpha(g_1g_2))})\mathrm{Ad}_{c(g_1,g_2)}T_{g_1}T_{g_2}\end{aligned}$$

which implies e) because of the last equation in Definition 5.2.  $\square$

Let  $\mathbf{a} = (a_1, \dots, a_n)$ . Define:

$$(6.2.14) \quad T_g\mathbf{a} = (T_g a_1, \dots, T_g a_n); \quad \mathrm{Ad}_c\mathbf{a} = (\mathrm{Ad}_c a_1, \dots, \mathrm{Ad}_c a_n);$$

and

$$(6.2.15) \quad \iota_a\mathbf{a} = \sum_{j=0}^n (-1)^{\sum_{i=1}^j (|a|+1)(|a_i|+1)} (a_1, \dots, a_j, a, a_{j+1}, \dots, a_n)$$

(for a homogeneous element  $a$ ).

By analogy with (6.2.12), (6.2.13), set

$$(6.2.16) \quad \mathcal{T}_g = \exp(\iota_{\alpha(g)})T_g$$

for every  $g \in G$ ;

$$(6.2.17) \quad \widetilde{\text{Ad}}(c) = \exp(-\iota_{D_c \cdot c^{-1}}) \text{Ad}(c)$$

for every  $c \in A^\times$  of degree zero.

Note that  $n$  could be equal to zero. In this case  $\iota_a(\mathbf{a}) = 0$ ,  $T_g \mathbf{a} = \mathbf{a}$ , and  $\text{Ad}(c)(\mathbf{a}) = \mathbf{a}$ .

For  $\mathbf{a}_1 = (a_1, \dots, a_{n_1})$ ,  $\mathbf{a}_2 = (a_{n_1+1}, \dots, a_{n_2})$ , etc., put

$$\varphi(\mathbf{a}_1, \mathbf{a}_2, \dots) = \varphi(a_1, \dots, a_{n_1}, a_{n_1+1}, \dots, a_{n_2}, \dots)$$

Every choice of  $n_1, \dots, n_{m+1} \geq 0$  such that  $n_1 + \dots + n_{m+1} = n$  defines a presentation  $(a_1, \dots, a_n) = (\mathbf{a}_1, \dots, \mathbf{a}_{m+1})$ . Define

$$|\mathbf{a}_k| = \sum_{i=n_k+1}^{n_{k+1}} |a_i|.$$

Put

$$(6.2.18) \quad (\phi(c_1, \dots, c_m)\varphi)(a_1, \dots, a_n) = \sum_{n_1, \dots, n_{m+1}} (-1)^{N(n_1, \dots, n_{m+1})}$$

$$\varphi(\mathbf{a}_1, c_1, \widetilde{\text{Ad}}_{c_1}^{-1} \mathbf{a}_2, c_2, \widetilde{\text{Ad}}(c_1 c_2)^{-1} \mathbf{a}_3, \dots, c_m, \widetilde{\text{Ad}}(c_1 c_2 \dots c_m)^{-1} \mathbf{a}_{m+1})(c_1 c_2 \dots c_m)^{-1}$$

Here

$$N(n_1, \dots, n_{m+1}) = \sum_{j=1}^m \sum_{i=1}^j (|\mathbf{a}_i| + n_i)$$

**Lemma 6.11.** *The operators  $\mathcal{T}_g$ ,  $\widetilde{\text{Ad}}(c)$ , and  $\phi(c_1, \dots, c_m)$  satisfy all the relations of Lemmas 6.1 and 6.2.*

*Proof.* Define for  $\mathbf{a} = (a_1, \dots, a_n)$  and for a homogenous derivation  $E$

$$(6.2.19) \quad E\mathbf{a} = \sum_{j=1}^n (-1)^{|E| \sum_{p < j} |a_p|} (a_1, \dots, E a_j, \dots, a_n)$$

Also put

$$(6.2.20) \quad \partial \mathbf{a} = \sum_{j=1}^{n-1} (-1)^{\sum_{p \leq j} |a_p|} (a_1, \dots, a_j a_{j+1}, \dots, a_n)$$

Note that Lemma 6.10 holds for  $\mathcal{T}_g$  and  $\widetilde{\text{Ad}}_c$  as in (6.2.16), (6.2.17) and for  $D$ ,  $\iota$ , etc. as above, if one replaces  $\delta$  by  $\partial$ . (In fact, a) can be easily checked, and the rest follows formally from a)). It is easy to deduce Lemma 6.11 from this.  $\square$

We get a generalization of Theorem 6.4:

**Theorem 6.12.** *There is an  $A_\infty$  action of  $G$  on  $C^\bullet(V, A, W)$  such that  $T(g)$  is equal to  $\mathcal{T}_g$  as in (6.2.12).*

**6.2.5. Behavior with respect to equivalences.** Now consider an equivalence between two actions up to inner automorphisms and compatible derivations

$$(6.2.21) \quad \mathbf{b} = (\{b(g)\}, \beta) : (T, c), (D, \alpha, R) \xrightarrow{\sim} (T', c'), (D', \alpha', R')$$

If  $V$  is a module with a derivation  $D_V$  and an action  $T_g$  compatible with the action on the left, let  $\mathbf{b}_*V$  be  $V$  equipped with the derivation  $D'_V$  and with the action  $T'_g$  compatible with the action on the right (cf. (5.3.2)). Let

$$\mathcal{B}_c = \mathcal{B}(A^\times, G \ltimes_c A^\times); \mathcal{B}_{c'} = \mathcal{B}(A^\times, G \ltimes_{c'} A^\times)$$

(cf. definitions in Lemma 6.3 and in Theorem 6.4).

**Lemma 6.13.** *The formulas*

$$g \mapsto b(g)g, g \in G; c \mapsto c, c \in A^\times$$

*define an isomorphism*

$$G \ltimes_c A^\times \xleftarrow{\sim} G \ltimes_{c'} A^\times$$

*of groups over  $G$ . Together with*

$$\Phi(c_1, \dots, c_m) \mapsto \Phi(c_1, \dots, c_m),$$

*they define an isomorphism of differential graded algebras*

$$\mathbf{b}^\dagger : \mathcal{B}_c \xleftarrow{\sim} \mathcal{B}_{c'}$$

*over  $k[G]$ .*

**Definition 6.14.**

$$\mathbf{b}^* = \exp(\iota_\beta) : C^\bullet(V, A, W) \xleftarrow{\sim} C^\bullet(\mathbf{b}_*V, A, \mathbf{b}_*W)$$

**Proposition 6.15.** *If one views  $C^\bullet(V, A, W)$  as a differential graded  $\mathcal{B}_{c'}$ -modules via the morphism  $\mathbf{b}^\dagger$ , then  $\mathbf{b}^*$  is a morphism of differential graded modules over  $\mathcal{B}_{c'}$ . For two composable equivalences  $\mathbf{b}_1$  and  $\mathbf{b}_2$ , one has*

$$(\mathbf{b}_1 \mathbf{b}_2)^\dagger = \mathbf{b}_2^\dagger \mathbf{b}_1^\dagger; (\mathbf{b}_1 \mathbf{b}_2)^* = \mathbf{b}_2^* \mathbf{b}_1^*$$

*Proof.* The statement follows from

$$\textbf{Lemma 6.16. } a) \widetilde{\text{Ad}}_{b(g)} \mathcal{T}_g \exp(\iota_\beta) = \exp(\iota_\beta) \mathcal{T}'_g$$

$$b) \widetilde{\text{Ad}}_c \exp(\iota_\beta) = \exp(\iota_\beta) \widetilde{\text{Ad}}'_c$$

To prove the lemma, observe

$$\widetilde{\text{Ad}}_{b(g)} \mathcal{T}_g \exp(\iota_\beta) \mathcal{T}'_g{}^{-1} =$$

$$\begin{aligned} & \exp(-\iota_{Db(g) \cdot b(g)^{-1}}) \text{Ad}_{b(g)} \exp(\iota_{\alpha(g)}) T_g \exp(\iota_\beta) \mathcal{T}'_g{}^{-1} \exp(-\iota_{\alpha'(g)}) = \\ & \exp(-\iota_{Db(g) \cdot b(g)^{-1}}) \exp(\iota_{\text{Ad}_{b(g)} \alpha(g)}) \exp(\iota_{\text{Ad}_{b(g)} T_g \beta}) \exp(-\iota_{\alpha'(g)}) = \exp(\iota_\beta) \end{aligned}$$

because of (5.2.1). This proves a). To prove b), note that

$$\begin{aligned} \widetilde{\text{Ad}}_c \exp(\iota_\beta) \widetilde{\text{Ad}}_c^{-1} &= \exp(-\iota_{Dc \cdot c^{-1}}) \text{Ad}_c \exp(\iota_\beta) \text{Ad}_c^{-1} \exp(\iota_{D'c \cdot c^{-1}}) = \\ & \exp(-\iota_{Dc \cdot c^{-1}}) \exp(\iota_{T_c \beta}) \exp(\iota_{Dc \cdot c^{-1} + \beta - T_c \beta}) = \exp(\iota_\beta) \end{aligned}$$



□

**6.2.6. Behavior with respect to Yoneda product.** Now let us describe the relation of the  $A_\infty$  action on a quotient to Yoneda product

$$(6.2.22) \quad \smile : C^\bullet(V_1, A, V_2) \otimes C^\bullet(V_2, A, V_3) \rightarrow C^\bullet(V_1, A, V_3)$$

given by

$$(6.2.23) \quad (\varphi \smile \psi)(a_1, \dots, a_{m+n}) = (-1)^{(|\varphi|+m)\sigma_j(|a_j|+1)} \varphi(a_1, \dots, a_m) \psi(a_{m+1}, \dots, a_{m+n})$$

**Lemma 6.17.** *The coproduct*

$$\Delta\phi(c_1, \dots, c_m) = \sum_{j=1}^m \phi(c_1, \dots, c_j) \otimes c_1 \dots c_j \phi(c_{j+1}, \dots, c_m)$$

turns the algebra  $\mathcal{B}_c$  into a differential graded bialgebra. The morphism (6.2.1) is a bialgebra morphism. If we write  $\Delta a = \sum a^{(1)} \otimes a^{(2)}$ , then

$$a(\varphi \smile \psi) = \sum a^{(1)} \varphi \smile a^{(2)} \psi$$

for  $a$  in  $\mathcal{B}_c$ . Morphisms  $b^\dagger$  from Lemma 6.13 are morphisms of bialgebras.

The proof is straightforward.

### 6.3. $A_\infty$ action on the standard complex: the case of Lie groupoids.

**6.3.1.  $A_\infty$  action of a Lie groupoid.** Consider a Lie groupoid  $\mathcal{G}$  with the manifold of objects  $M$ . Let  $\mathcal{A}^\bullet$  be a sheaf of  $\mathcal{O}_M^\bullet$ -algebras with an action of  $\mathcal{G}$  up to inner automorphisms and with a compatible flat connection up to inner derivations as in 5.7.2. Recall the presheaves  $\underline{\mathcal{G}}^{(n)}$  on  $M^{n+1}$  (5.7.1). Let also

$$(6.3.1) \quad \underline{\mathcal{G}}_{jk}^{(n)} = p_{jk}^{-1} \underline{\mathcal{G}}$$

where  $p_{jk} : M^{n+1} \rightarrow M^2$  is the projection to the  $j$ th and  $k$ th components.

**Definition 6.18.** An  $A_\infty$  action of  $\mathcal{G}$  on a differential graded  $\mathcal{O}_M^\bullet$ -module  $\mathcal{C}^\bullet$  is a collection of morphisms

$$T : \mathcal{G}^{(n)} \rightarrow \underline{\mathbf{Hom}}^{1-n}(p_{n+1}^* \mathcal{C}^\bullet, p_1^* \mathcal{C}^\bullet),$$

$n \geq 1$ , such that (6.1.1) holds for every  $g_1, \dots, g_n$  where  $g_j$  is a local section of  $\underline{\mathcal{G}}_{j,j+1}^{(n)}$ .

An  $A_\infty$  morphism of  $A_\infty$  actions is a collection of morphisms

$$\phi : \underline{\mathcal{G}}^{(n)} \rightarrow \underline{\mathbf{Hom}}^{-n}(p_{n+1}^* \mathcal{C}^\bullet, p_1^* \mathcal{C}^\bullet),$$

$n \geq 0$ , such that (6.1.2) holds.

**6.3.2. Action on the standard complex.** Let  $\mathcal{V}^\bullet$  and  $\mathcal{W}^\bullet$  be two graded  $\mathcal{A}^\bullet$ -modules with compatible actions of  $\mathcal{G}$  and with compatible connections  $\nabla$ . Sometimes, to distinguish, we denote the three connections by  $\nabla_{\mathcal{A}}$ ,  $\nabla_{\mathcal{V}}$ , and  $\nabla_{\mathcal{W}}$  respectively. Compatibility means, as usual, that

$$\nabla(av) = \nabla(a)v + (-1)^{|a|}a\nabla(v)$$

for  $a \in \mathcal{A}^\bullet$  and  $v \in \mathcal{V}^\bullet$ .

**Definition 6.19.** The standard complex  $\mathcal{C}^\bullet(\mathcal{V}^\bullet, \mathcal{A}^\bullet, \mathcal{W}^\bullet)$  is the complex of sheaves

$$\mathcal{C}^m = \prod_{p+n=m} \underline{\mathrm{Hom}}_{\mathcal{O}_M^\bullet}^p(\otimes_{\mathcal{O}_M^\bullet}^n \mathcal{A}^\bullet, \underline{\mathrm{Hom}}_{\mathcal{O}_M^\bullet}(\mathcal{V}^\bullet, \mathcal{W}^\bullet))$$

with the differential  $\delta + \nabla + \iota_R$  (cf. (6.2.10), (6.2.11), and Corollary 6.8).

*Remark 6.20.* In other words,  $\mathcal{C}^\bullet$  is the standard complex computed over the algebra of scalars  $\mathcal{O}_M^\bullet$  and sheafed. An example arises when  $\mathcal{A}$  is a bundle of algebras with a flat connection,  $\mathcal{V}$  and  $\mathcal{W}$  are bundles of modules with compatible flat connections,  $\mathcal{O}_M$  is the differential graded algebra of forms, and  $\mathcal{V}^\bullet$ , resp.  $\mathcal{W}^\bullet$ , is the module of  $\mathcal{V}$ - (resp.  $\mathcal{W}$ )-valued forms. In this case  $\mathcal{C}^\bullet(\mathcal{V}, \mathcal{A}, \mathcal{W})$  is a bundle of complexes with an induced flat connection, and  $\mathcal{C}^\bullet(\mathcal{V}^\bullet, \mathcal{W}^\bullet)$  is the complex of forms with values in this bundle. Our situation is different in only one regard. Namely, our  $\mathcal{O}^\bullet$  will be mainly the algebra of  $\Lambda$ -valued forms. Accordingly, the exact nature of local cochains  $\varphi(a_1, \dots, a_n; v)$  that we allow needs to be specified. We will do this in 8.1.

**Theorem 6.21.** There is an  $A_\infty$  action of  $\mathcal{G}$  on  $\mathcal{C}^\bullet(\mathcal{V}^\bullet, \mathcal{A}^\bullet, \mathcal{W}^\bullet)$  such that  $T(g)$  is equal to  $\mathcal{T}_g$  as in (6.2.12).

*Proof.* The operators  $T(g_1, \dots, g_n)$  are computed by a recursive procedure from Remark 6.5 where  $\phi(c_1, \dots, c_n)$  are as in (6.2.18). The only difference is that the morphism (6.2.8) sends  $c$  not to  $\mathrm{Ad}(c)$  but to  $\widetilde{\mathrm{Ad}}(c)$  (cf. (6.2.12), (6.2.13)).  $\square$

**6.4. The cochain complex of an  $A_\infty$  action.** Given a sheaf of  $\mathcal{O}_M^\bullet$ -modules  $\mathcal{M}^\bullet$  with an  $A_\infty$  action of a Lie groupoid  $\mathcal{G}$ , define

$$\mathcal{C}^\bullet(M, \mathcal{M}^\bullet) = \prod_{n=0}^{\infty} \Gamma(M^{n+1}, \underline{\mathrm{Hom}}(\underline{\mathcal{G}}^{(n)}, p_1^* \mathcal{M}^{\bullet-n}))$$

with the differential

$$\begin{aligned} (d\Phi)(g_1, \dots, g_{n+1}) &= \nabla_{\mathcal{M}}\Phi(g_1, \dots, g_{n+1}) + \sum_{j=1}^n T(g_1, \dots, g_j)\Phi(g_{j+1}, \dots, g_{n+1}) + \\ &+ \sum_{j=1}^n (-1)^j \Phi(g_1, \dots, g_j g_{j+1}, \dots, g_{n+1}) + (-1)^{n+1} \Phi(g_1, \dots, g_n) \end{aligned}$$

Here  $g_j$  is a local section of  $\underline{\mathcal{G}}_{j,j+1}^{(n)}$ , cf. (6.3.1).

## 7. THE $A_\infty$ ACTION OF $\pi_1(M)$ ON STANDARD COMPLEXES OF $\mathcal{A}_M^\bullet$ -MODULES

**7.1. The action of  $\pi_1(M)$  up to inner automorphisms on  $\mathcal{A}_M^\bullet$ .** Assume that  $M$  is a symplectic manifold with a chosen  $\mathrm{Sp}^4$  structure. In this section we construct:

1) a groupoid  $\tilde{\mathbf{G}}_M$  together with an epimorphism  $\tilde{\mathbf{G}}_M \rightarrow \pi_1(M)$  and a morphism of groups

$$(7.1.1) \quad \mathrm{Ker}(\tilde{\mathbf{G}}_{x,x} \xrightarrow{p} \pi_1(M)_{x,x}) \xrightarrow{i} \mathcal{A}_{M,x}^\times;$$

2) an action of  $\tilde{\mathbf{G}}_M$  on  $\mathcal{A}_M$  up to inner automorphisms such that any element  $h$  of  $\mathrm{Ker}(p)$  acts by conjugation with  $i(h)$ ;

3) a flat connection on  $\mathcal{A}_M$  up to inner derivations compatible with the action of  $\tilde{\mathbf{G}}_M$ , such that  $\nabla$  is a Fedosov connection  $\nabla_{\mathcal{A}}$  with curvature  $\frac{1}{\hbar}\omega$ .

A more straightforward construction works in general under the assumption that  $M$  has an  $\mathrm{Sp}^4$  structure and yields the connection with  $R = \frac{1}{i\hbar}\omega$ . A construction that is a little more involved yields a connection with  $R = 0$  under an additional restriction:

$$(7.1.2) \quad \langle \pi_2(M), [\omega] \rangle = 0$$

meaning that the class of the symplectic form vanishes on the image of the Hurewicz homomorphism.

By Lemma 5.6 we will conclude that

**Proposition 7.1.** *The sheaf of algebras*

$$(7.1.3) \quad \mathcal{A}_M^\bullet = \Omega_M^\bullet(\mathcal{A})$$

*of  $\mathcal{A}_M$ -valued forms on  $M$  carries an action of  $\pi_1(M)$  up to inner automorphisms and a compatible flat connection up to inner derivations such that  $\nabla$  is a Fedosov connection  $\nabla_{\mathcal{A}}$  with curvature  $\frac{1}{\hbar}\omega$ .*

Now Theorem 6.21 implies

**Theorem 7.2.** *For any two differential graded  $\mathcal{A}_M^\bullet$ -modules  $\mathcal{V}^\bullet, \mathcal{W}^\bullet$  with a compatible action of  $\pi_1(M)$  and a compatible connection, the standard complex  $\mathcal{C}^\bullet(\mathcal{V}^\bullet, \mathcal{A}^\bullet, \mathcal{W}^\bullet)$  has a natural  $A_\infty$  action of  $\pi_1(M)$ .*

**7.2. The construction of the groupoid  $\tilde{\mathbf{G}}_M$ .** There are two options for constructing the groupoid  $\tilde{\mathbf{G}}_M$  and a flat connection up to inner derivations.

**7.2.1. The connection with  $R = \frac{1}{i\hbar}\omega$ .** Assume that  $M$  is a symplectic manifold with an  $\mathrm{Sp}^4$  structure. Let  $g_{jk}$  be an  $\mathrm{Sp}^4(2n, \mathbb{R})$ -cocycle whose projection to  $\mathrm{Sp}$  is a cocycle representing the tangent bundle. Consider the groupoid of the bundle represented by the cocycle  $g$  (viewed as a twisted bundle with  $c=1$ ). Here the role of  $G$  (as in 14.2) is played by the group  $\mathrm{Sp}^4(2n)$  as in 12.

Consider a lifted Fedosov connection with curvature  $\frac{1}{i\hbar}\omega$  (cf. Theorem 3.5). This is a partial case of a connection defined in 14.2.2. Now we can

define a flat connection up to inner derivations as in 14.2.2. (Observe that  $\frac{1}{i\hbar}\mathbb{A}$  is a Lie subalgebra of the associative algebra  $\mathcal{A}$  and  $\mathrm{Sp}^4(2n, \mathbb{R})$  is a subgroup of  $\mathcal{A}^\times$ ).

**7.2.2. The connection with  $R = 0$ .** Consider the cocycle  $g_{jk}$  as above in 7.2.1. Consider  $\tilde{g}_{jk} \in \exp(\frac{1}{i\hbar}\mathbb{R})$  defined by

$$(7.2.1) \quad \tilde{g}_{jk} = \exp\left(\frac{1}{i\hbar}f_{jk}\right)$$

where

$$(7.2.2) \quad \omega|U_j = d\alpha_j; \quad \alpha_j - \alpha_k = df_{jk}$$

Observe that

$$(7.2.3) \quad c_{jkl} = \exp\left(\frac{1}{i\hbar}(f_{jk} + f_{kl} - f_{jl})\right)$$

takes values in  $\exp(\frac{1}{i\hbar}\mathbb{R})$  and represents the class  $\exp(\frac{1}{i\hbar}[\omega])$ . If our lifted Fedosov connection is represented by a collection of  $\tilde{\mathfrak{g}}$ -valued one-forms  $A_j$ , then

$$(7.2.4) \quad \tilde{A}_j = \frac{1}{i\hbar}\alpha_j + A_j$$

represents a *flat* connection in the twisted bundle given by  $\tilde{g}_{jk}, c_{jkl}$ . Now we can define a flat connection up to inner derivations exactly as we did in 7.2.1. Now we have  $R = 0$ .

There is a short exact sequence of groups

$$(7.2.5) \quad 1 \rightarrow \mathrm{Sp}^4(2n, \mathbb{R}) \rightarrow (\tilde{\mathbf{G}}_M)_{x,x} \rightarrow \pi_1(M, x) \rightarrow 1$$

for any point  $x$  of  $M$ .

## 8. RESUMÉ OF THE GENERAL PROCEDURE

We summarize the construction that we described up to this point. This includes the notions of objects and the construction of the infinity local system of morphisms between two objects. Next (in section 9.1) we will present a construction of a special type of objects.

**8.1.  $\Omega_{\mathbb{K}, M}^\bullet$ -modules and their inverse images.** Recall the definition of the sheaf  $\Omega_{\mathbb{K}, M}^\bullet$  of  $\mathbb{K}$ -valued forms on a manifold  $M$  (Definition 1.1). We will be considering the following class of sheaves of  $\Omega_{\mathbb{K}, M}^\bullet$ -modules. Start with a vector bundle  $E$  (finite or profinite) and a fiber bundle  $\mathfrak{X}$  on  $M$ . Local sections of the module  $\mathcal{M}_{\mathcal{E}, \mathfrak{X}}^\bullet$  are countable sums

$$(8.1.1) \quad \sum_{\varphi, \Phi} a_{\Phi, \varphi} \exp\left(\frac{1}{i\hbar}\varphi\right) e_\Phi$$

where  $a_{\Phi,\varphi}$  are local differential forms with coefficients in  $E$ ,  $\varphi$  are local sections of  $C_M^\infty$ ,  $e_\Phi$  are formal symbols corresponding to local sections  $\Phi$  of  $\mathfrak{X}$ , and  $\varphi \rightarrow +\infty$ . For a smooth map  $M \rightarrow N$  we define

$$(8.1.2) \quad f^* \mathcal{M}_{\mathfrak{E},\mathfrak{X}}^\bullet = \mathcal{M}_{f^*E, f^*\mathfrak{X}}^\bullet$$

We consider the following differentials on  $\mathcal{M}_{E,\mathfrak{X}}^\bullet$ . Let  $E_0$  be a fiber of  $E$  and let  $X$  be a fiber of  $\mathfrak{X}$ . Choose any local trivialization of the bundles  $E$  and  $\mathfrak{X}$  near  $x_0$ . Also choose any local coordinate systems on  $M$  near  $x_0$  and on  $\mathfrak{X}$  near  $\Phi(x_0)$ . Then we can identify local sections of  $E$  with local functions  $M \rightarrow E_0$  and local sections of  $\mathfrak{X}$  with local maps  $M \rightarrow \mathbb{R}^{\dim \mathfrak{X}}$ . We require the differential to be of the form

$$(8.1.3) \quad \begin{aligned} \nabla_{\mathcal{M}} \sum_{\varphi, \Phi} a_{\Phi,\varphi} \exp\left(\frac{1}{i\hbar} \varphi\right) e_\Phi &= \sum_{\varphi, \Phi} da_{\Phi,\varphi} \exp\left(\frac{1}{i\hbar} \varphi\right) e_\Phi + \\ &+ \sum_{\varphi, \Phi} \frac{1}{i\hbar} \varphi' a_{\Phi,\varphi} \exp\left(\frac{1}{i\hbar} \varphi\right) e_\Phi dx + \sum_{\varphi, \Phi} A(x, \Phi(x)) \Phi'(x) a_{\Phi,\varphi} \exp\left(\frac{1}{i\hbar} \varphi\right) e_\Phi dx + \\ &+ \sum_{\varphi, \Phi} B(x, \Phi(x)) a_{\Phi,\varphi} \exp\left(\frac{1}{i\hbar} \varphi\right) e_\Phi \end{aligned}$$

Here  $A$  and  $B$  are local  $\text{End}(E_0)$ -valued functions on  $M \times X$ . If  $\nabla_{\mathcal{M}}$  is of the above form for one choice of the local trivializations then it is true for any such choice. We will use the shorthand

$$(8.1.4) \quad \nabla_{\mathcal{M}} e_\Phi = (A\Phi' + B)e_\Phi$$

Now  $f : M \rightarrow N$  is a smooth map. A differential  $\nabla_{\mathcal{M}}$  on  $\mathcal{M}_{E,\mathfrak{X}}^\bullet$  induces a differential  $f^* \nabla_{\mathcal{M}}$  on  $f^* \mathcal{M}_{E,\mathfrak{X}}^\bullet = \mathcal{M}_{f^*E, f^*\mathfrak{X}}^\bullet$  as follows. Let  $x$  be local coordinates on  $M$ ,  $y$  local coordinates on  $N$ , and let the map be locally of the form  $y = f(x)$ . If

$$\nabla_{\mathcal{M}} e_\Psi = (A(y, \Psi(y)) \Psi'(y) dy + B(y, \Psi(y)) dy) e_\Psi$$

for any  $\Psi$ , then

$$f^* \nabla_{\mathcal{M}} e_\Phi = (A(f(x), \Phi(x)) \Phi'(x) dx + B(f(x), \Phi(x)) f'(x) dx) e_\Phi$$

In other words: let  $p : \mathfrak{X} \rightarrow M$  be the projection. Locally in  $\mathfrak{X}$  (near  $\Phi(x)$ ), we require that there exist linear operators  $A(z) : T_z \mathfrak{X}_{p(z)} \rightarrow \text{End } E_{p(z)}$  and  $B(z) : T_{p(z)} M \rightarrow \text{End } E_{p(z)}$  and a linear projection  $P(z) : T_z \mathfrak{X} \rightarrow T_z \mathfrak{X}_{p(z)}$ , all smoothly depending on  $z \in \mathfrak{X}$ , such that for any point  $x$  of  $M$  and for any  $\eta \in T_x M$ ,

$$(8.1.5) \quad \nabla_{\mathcal{M}} e_\Phi(x)(\eta) = (A(\Phi(x)) P(d\Phi(x)) \eta + B(\Phi(x)) \eta) e_\Phi$$

Note that if  $\nabla_{\mathcal{M}}$  satisfies this property for one choice of  $P$  then it satisfies it for any other choice. This is because for any two projections  $P_1$  and  $P_2$ ,  $(P_1 - P_2) d\Phi(x) : T_x M \rightarrow T_{\Phi(x)} \mathfrak{X}_{\Phi(x)}$  is a linear operator depending only on the value  $\Phi(x)$ .

For  $f : M \rightarrow N$ , if  $\nabla_{\mathcal{M}}$  is locally determined by  $A(z)$ ,  $B(z)$ , and  $P(z)$ , so is  $f^* \nabla_{\mathcal{M}}$ .

**8.2. Oscillatory modules.** Consider the bundle  $\mathcal{A}_M^\bullet$  with the action of the groupoid  $\tilde{\mathbf{G}}_M$  up to inner automorphisms and a compatible flat connection up to inner derivations as defined in 7.2.2. By definition, an oscillatory module  $\mathcal{V}^\bullet$  is a graded module over  $\mathcal{A}_M^\bullet$  of the type defined in 8.1, with a compatible action of the groupoid  $\tilde{\mathbf{G}}_M$  and a compatible flat connection as in 5.8.

**8.3.  $\Omega_{\mathbb{K},M}^\bullet$ -modules with  $\pi_1$ -action.** These modules are defined in 6.3.1 (in our case here,  $\mathcal{O}_M^\bullet = \Omega_{\mathbb{K},M}^\bullet$  as in Definition 1.1). More generally, *twisted*  $(\Omega_{\mathbb{K},M}^\bullet, \pi_1(M))$  modules are defined in 16.3. By Theorem 6.21 and Lemma 5.6, under the assumptions  $c_1(M) = 0$  and (7.1.2), the standard complex (Definition 6.19) of two oscillatory modules is a twisted  $\Omega_{\mathbb{K},M}^\bullet$ -module with  $\pi_1$ -action. We denote this complex by  $\mathcal{C}^\bullet(\mathcal{V}^\bullet, \mathcal{A}^\bullet, \mathcal{W}^\bullet)$ .

**8.4. Infinity local systems of  $\mathbb{K}$ -modules.** An infinity local systems of  $\mathbb{K}$ -modules on a manifold  $X$  is a collection of complexes of  $\mathbb{K}$ -modules  $\mathcal{C}_x^\bullet$ ,  $x \in X$ , together with linear maps

$$(8.4.1) \quad T(g_1, \dots, g_n) : \mathcal{C}_{x_{n+1}}^\bullet \rightarrow \mathcal{C}_{x_1}^{\bullet+1-n}$$

for any  $g_j \in \pi_1(X)_{x_j, x_{j+1}}$ ,  $j = 1, \dots, n$ , subject to (6.1.1). In other words, this is an  $A_\infty$  action of the fundamental groupoid  $\pi_1(X)$ , cf. 6.2.3.

**8.4.1. From twisted  $(\Omega_{\mathbb{K},M}^\bullet, \pi_1(M))$  modules to infinity local systems.** If  $\mathcal{M}^\bullet$  is an  $\Omega_{\mathbb{K},M}^\bullet$ -module with a twisted  $\pi_1$ -action (as in 8.3), then

$$(8.4.2) \quad \mathcal{C}_x^\bullet = \varinjlim_{x \in U} C^\bullet(U, \mathcal{M}^\bullet)$$

is an infinity local system of  $\mathbb{K}$ -modules. (cf. 6.4 for the definition of the cochain complex  $C^\bullet(U, \mathcal{M}^\bullet)$ ). This is explained in detail in 16.3.2.

**Definition 8.1.** Given two oscillatory modules  $\mathcal{V}^\bullet$  and  $\mathcal{W}^\bullet$  on a symplectic manifold  $M$  that has an  $\mathrm{Sp}^4$  structure and satisfy (7.1.2), we denote by  $\underline{\mathbb{R}\mathrm{HOM}}(\mathcal{V}^\bullet, \mathcal{W}^\bullet)$  the complex  $\mathcal{C}^\bullet$  (cf. 8.4) constructed from the complex  $\mathcal{M}^\bullet = \mathcal{C}^\bullet(\mathcal{V}^\bullet, \mathcal{A}^\bullet, \mathcal{W}^\bullet)$  (cf. 8.3).

## 9. OBJECTS CONSTRUCTED FROM LAGRANGIAN SUBMANIFOLDS

### 9.1. Induced modules.

**9.1.1. The case of groups acting on algebras.** Let  $i : B \rightarrow A$  be a morphism of algebras and let  $j : P \rightarrow G$  be a morphism of groups. Assume that  $P$  acts on  $B$  by automorphisms and  $G$  acts on  $A$  by automorphisms. We denote these automorphisms by  $S_p$ ,  $p \in P$ , and  $T_g$ ,  $g \in G$ . We assume that  $i(T_p b) = T_{jp}(ib)$  for any  $p$  and  $g$ . For simplicity, we consider here only true actions, i.e. those for which  $c(g_1, g_2) = 1$  and  $c(p_1, p_2) = 1$ .

Let  $W$  be a  $B$ -module with a compatible action of  $P$  denoted by  $S_p : W \xrightarrow{\sim} W, p \in P$ . Define the induced module  $V$  as follows. First consider the  $A$ -module  $A \otimes_B W$ . Note that it carries a compatible action of  $P$  :

$$(9.1.1) \quad S_p(a \otimes w) = T_{jp}(a)S_p(w).$$

Now let  $V$  be the quotient of the space of formal linear combinations

$$(9.1.2) \quad \sum_{g \in G} T_g v_g, \quad v_g \in A \otimes_B W,$$

by the linear span of  $T_{gj(p)}(a \otimes w) - T_g S_p(a \otimes w), g \in G, p \in P, a \in A, w \in W$ . Define the  $A$ -module structure on  $V$  by

$$(9.1.3) \quad a \sum T_g v_g = \sum T_g (T_g^{-1} a) v_g$$

and a compatible group action of  $G$

$$(9.1.4) \quad T_{g_0} \sum T_g v_g = \sum T_{g_0 g} v_g$$

This is just another way of defining the induced module

$$(9.1.5) \quad V = (G \ltimes A) \otimes_{P \ltimes B} W$$

Now assume that  $A$  and  $B$  are graded algebras. Let  $\{D : A \rightarrow A; \alpha(g)|g \in G; R_A\}$  and  $\{E : B \rightarrow B; \beta(g)|g \in B; R_B\}$  be derivations of square zero of  $A$  and of  $B$  up to inner derivations. We assume that these derivations are compatible with  $i$  and  $j$ , *i.e.*

$$(9.1.6) \quad i(E(b)) = D(i(b)); i(\beta(p)) = \alpha(jp); i(R_B) = R_A.$$

Let  $E_W : W \rightarrow W$  be a compatible derivation of  $W$ . Then  $A \otimes_B W$  carries a derivation  $E_{A \otimes_B W}$  compatible with the action of  $B$ ;

$$(9.1.7) \quad E_{A \otimes_B W}(a \otimes w) = D_A(a) \otimes w + (-1)^{|a|} a \otimes E_W(w).$$

This allows to define a derivation of the induced module  $V$  compatible with the action of  $G$ ;

$$(9.1.8) \quad D_V(\sum T_g v_g) = \sum T_g (\alpha(g^{-1}) v_g) + \sum T_g E_{A \otimes_B W}(v_g)$$

**9.1.2. The case of groupoids.** Now generalize the situation of 9.1.1 to the case when  $P$  is a groupoid with the set of objects  $Y$  and  $G$  is a groupoid with the set of objects  $X$ . Denote by  $j : Y \rightarrow X$  the action of the morphism of groupoids  $j$  on objects. In this case  $A = \{A_x | x \in X\}$ ,  $B = \{B_y | y \in Y\}$ , and  $W = \{W_y | y \in Y\}$ . Put

$$(9.1.9) \quad (A \otimes_B W)_y = A_{jy} \otimes_{B_y} W_y$$

Formulas (9.1.1) and (9.1.7) define a compatible action of  $P$  and a compatible derivation on  $A \otimes_B W$ .

$$(9.1.10) \quad V_x = \left\{ \sum_{y \in Y, g \in G_{x, jy}} T_g v_g | v_g \in (A \otimes_B W)_y \right\} / \langle T_{gj(p)} v - T_g (S_p v) \rangle$$

Formulas (9.1.3), (9.1.4), (9.1.7), and (9.1.8) define on  $V$  an  $A$ -module structure, a compatible action of  $G$ , and a compatible derivation.

**9.1.3. The case of Lie groupoids.** Now let  $\mathcal{G}$  and  $\mathcal{P}$  be Lie groupoids with the manifolds of objects  $X$  and  $Y$  respectively. Let  $j : \mathcal{P} \rightarrow \mathcal{G}$  be a morphism of Lie groupoids, *i.e.* a smooth map  $X \rightarrow Y$  and a smooth map  $\mathcal{P} \rightarrow \mathcal{G}$  over  $X \times X$  that preserves the composition and the unit. Let  $\mathcal{B}^\bullet$  be a sheaf of  $\mathcal{O}_Y^\bullet$ -algebras and let  $\mathcal{A}^\bullet$  be a sheaf of  $\mathcal{O}_X^\bullet$ -algebras, together with a morphism  $i : \mathcal{B}^\bullet \rightarrow j^* \mathcal{A}^\bullet$ . Consider an action  $S$  of  $\mathcal{P}$  on  $\mathcal{B}^\bullet$  and an action  $T$  of  $\mathcal{G}$  on  $\mathcal{A}^\bullet$ . We assume that the morphism  $i$  preserves the action of  $\mathcal{P}$ . Furthermore, let  $(\nabla_{\mathcal{B}}, \beta, R_{\mathcal{B}})$  be a compatible flat connection up to inner derivations on  $\mathcal{B}^\bullet$  and let  $(\nabla_{\mathcal{A}}, \alpha, R_{\mathcal{A}})$  be a compatible flat connection up to inner derivations on  $\mathcal{A}^\bullet$ . We require the following compatibility conditions generalizing (9.1.6):

$$(9.1.11) \quad i(\nabla_{\mathcal{B}} b) = (j^* \nabla_{\mathcal{A}})(ib)$$

in  $j^* \mathcal{A}^\bullet$  on  $Y$ , for any local section  $b$  of  $\mathcal{B}^\bullet$ ;

$$(9.1.12) \quad i(R_{\mathcal{B}}) = j^*(R_{\mathcal{A}})$$

in  $j^* \mathcal{A}^\bullet$ ; and

$$(9.1.13) \quad i(\beta(p)) = \alpha(jp)$$

in  $j^* \mathcal{A}^\bullet$  for any local section  $p$  of  $\mathcal{P}$ .

*Remark 9.1.* The latter equation requires some explanation. It is not *a priori* clear why, for a local section  $g$  of  $\mathcal{G}$ ,  $\alpha(g)$  depends only on the restriction of  $g$  to  $Y \times Y$ . To ensure this, we will always assume that the form  $\alpha(g)$  is obtained from a local section  $g$  by the same procedure as the factor in front of  $e_\Phi$  in the right hand side of (8.1.4) is obtained from a local section  $\Phi$ .

Now assume that  $\mathcal{W}^\bullet$  is a  $\mathcal{B}^\bullet$ -module with a compatible action  $S$  of  $\mathcal{P}$  and a compatible connection  $\nabla_{\mathcal{W}}$ . The module  $j^* \mathcal{A}^\bullet \otimes_{\mathcal{B}^\bullet} \mathcal{W}$  has a compatible action  $S$  of  $\mathcal{P}$  and a compatible connection  $\nabla_{\mathcal{A} \otimes_{\mathcal{B}} \mathcal{W}}$  given by

$$(9.1.14) \quad S_p(a \otimes w) = T_{jp}(a) \otimes S_p w;$$

$$(9.1.15) \quad \nabla_{\mathcal{A} \otimes_{\mathcal{B}} \mathcal{W}}(a \otimes w) = \nabla_{\mathcal{A}}(a) \otimes w + (-1)^{|a|} a \otimes \nabla_{\mathcal{W}}(w)$$

Now define the induced module  $\mathcal{V}$  as follows. First, for any open subsets  $U$  of  $X$  and  $U'$  of  $Y$  and any smooth map  $f : U \rightarrow U'$ , let  $\underline{\mathcal{G}}_f$  be the inverse image of  $\underline{\mathcal{G}}$  under

$$(9.1.16) \quad U \xrightarrow{\sim} \text{graph}(f) \hookrightarrow X \times Y \hookrightarrow X \times X$$

The space of local sections of  $\mathcal{V}^\bullet$  over  $U$  is the space of formal linear combinations

$$(9.1.17) \quad \sum_{U, U'} \sum_{f: U \rightarrow U'} \sum_{g \in \underline{\mathcal{G}}_f(U)} T_g v_g; \quad v_g \in (\mathcal{A}^\bullet \otimes_{\mathcal{B}^\bullet} \mathcal{W}^\bullet)(U')$$



factorized by the linear span of

$$(9.1.18) \quad T_{gj(p)}(a \otimes w) - T_g(S_p(a \otimes w))$$

for some  $h : U' \rightarrow U''$ ,  $f : U \rightarrow U'$ ,  $g$  a local section of  $\underline{\mathcal{G}}_f(U)$ , and  $p$  a local section of  $\underline{\mathcal{P}}|_{\text{graph}(h)}$ . We interpret  $gj(p)$  as a local section of  $\underline{\mathcal{G}}_{hf}$ .

Formulas

$$(9.1.19) \quad T_{g_0} \sum T_g v_g = \sum T_{g_0 g} v_g; \quad a \sum T_g v_g = \sum T_g (T_{g^{-1}}(a) v_g);$$

$$(9.1.20) \quad \nabla_{\mathcal{V}} \sum T_g v_g = \sum T_g \alpha(g^{-1}) v_g + T_g \nabla_{\mathcal{A} \otimes_{\mathcal{B}} \mathcal{W}}(v_g)$$

define an  $\mathcal{A}^\bullet$ -module structure, a compatible action of  $\mathcal{G}$ , and a compatible connection on  $\mathcal{V}^\bullet$ . Note that the last formula relies again on the assumption discussed in Remark 9.1. Indeed, we need to be sure that  $\alpha(g^{-1})|_{\text{graph}(f)}$  depends only on  $g|_{\text{graph}(f)}$ .

**9.1.4. General definition of an induced module.** Finally, let us assume, analogously to what we did in 5.8.1, that there is a Lie groupoid  $\Gamma$  on  $X$  and a Lie groupoid  $\Pi$  on  $Y$  together with a morphism  $\Pi \rightarrow j^* \Gamma$ , an epimorphism  $\mathcal{G} \rightarrow \Gamma$ , and an epimorphism  $\mathcal{P} \rightarrow \Pi$  such that the diagram

$$\begin{array}{ccc} \mathcal{P} & \longrightarrow & \Pi \\ \downarrow & & \downarrow \\ j^* \mathcal{G} & \longrightarrow & j^* \Gamma \end{array}$$

commutes. Let  $\mathcal{H}_x = \text{Ker}(\mathcal{G}_{x,x} \rightarrow \Gamma_{x,x})$  and  $\mathcal{Q}_y = \text{Ker}(\mathcal{P}_{y,y} \rightarrow \Pi_{y,y})$ . Denote by  $\underline{\mathcal{H}}$ , resp.  $\underline{\mathcal{Q}}$ , the sheaf of sections of the bundle of groups  $\mathcal{H}$ , resp.  $\mathcal{Q}$ . We also assume that there are morphisms of sheaves  $i : \underline{\mathcal{H}} \rightarrow \mathcal{A}^\times$  and  $i : \underline{\mathcal{Q}} \rightarrow \mathcal{B}^\times$  such that the diagram

$$\begin{array}{ccc} \underline{\mathcal{Q}} & \longrightarrow & \mathcal{A}^\times \\ \downarrow & & \downarrow \\ j^* \underline{\mathcal{H}} & \longrightarrow & j^* \mathcal{B}^\times \end{array}$$

commutes. We also assume that the  $\mathcal{B}^\bullet$ -module  $\mathcal{W}^\bullet$  and the flat connection up to inner derivations  $(\nabla_{\mathcal{B}}, \beta, R_{\mathcal{B}})$  satisfies

$$S_q w = i(q)w; \quad \beta(q) = -\nabla_{\mathcal{B}} i(q) \cdot (iq)^{-1}$$

for any local sections  $q$  of  $\underline{\mathcal{Q}}$  and  $w$  of  $\mathcal{W}^\bullet$ .

**Definition 9.2.** Under the assumptions above, the induced module is the quotient of the module  $\mathcal{V}^\bullet$  ((9.1.19), (9.1.20)) by the submodule generated by elements  $T_h v - i(h)v$ ,  $h$  being any local section of  $\mathcal{H}$  and  $v$  any local section of  $\mathcal{V}^\bullet$ .

**9.2. The induced oscillatory module  $\mathcal{V}_L$ .**

**9.2.1. The algebra  $\mathcal{B}$  and the module  $\widehat{\widehat{\mathbb{V}}}_{\mathbb{K}}$ .** Recall the grading

$$(9.2.1) \quad |\widehat{x}_j| = |\widehat{\xi}_j| = 1; \quad |\hbar| = 2$$

Now define

$$(9.2.2) \quad \widehat{\mathbb{V}} = \mathbb{C}[[\hbar]]; \quad \widehat{\widehat{\mathbb{V}}} = \left\{ \sum_{k=-N}^{\infty} v_k | v_k \in \widehat{\mathbb{V}}[\hbar^{-1}]_k \right\}$$

where  $N$  runs through all integers.

**Definition 9.3.** Put

$$(9.2.3) \quad \widehat{\widehat{\mathbb{V}}}_{\mathbb{K}} = \left\{ \sum_{k=0}^{\infty} e^{\frac{1}{i\hbar} c_k} v_k | \in \widehat{\widehat{\mathbb{V}}}; \quad c_k \in \mathbb{R}; \quad c_k \rightarrow \infty \right\}$$

$$(9.2.4) \quad \widehat{\widehat{\mathbb{V}}}_{\Lambda} = \left\{ \sum_{k=0}^{\infty} e^{\frac{1}{i\hbar} c_k} v_k | \in \widehat{\widehat{\mathbb{V}}}; \quad c_k \geq 0; \quad c_k \rightarrow \infty \right\}$$

Now define the subalgebra  $\mathcal{B}$  of  $\mathcal{A}$  (Definition 4.2) by

$$(9.2.5) \quad \mathcal{B} = \text{MPar}(n) \ltimes \widehat{\widehat{\mathbb{A}}}$$

(cf. 12.1).

**Lemma 9.4.** *The formulas*

$$\begin{aligned} \widehat{x} &\mapsto \widehat{x}; \quad \widehat{\xi} \mapsto i\hbar \frac{\partial}{\partial \widehat{x}}; \\ \begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix} &\mapsto T_b, \quad (T_b f)(x) = \frac{1}{\sqrt{\det(b)}} f(b^{-1}x); \\ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} &\mapsto \exp\left(-\frac{i\hbar}{2} a \frac{\partial^2}{\partial \widehat{x}}\right) \end{aligned}$$

define an action of  $\text{MPar}(n)$  that together with the action of  $\widehat{\widehat{\mathbb{A}}}$  turns  $\widehat{\widehat{\mathbb{V}}}_{\mathbb{K}}$  into a  $\mathcal{B}$ -module.

**Definition 9.5.**

$$\widehat{\mathcal{V}} = \mathcal{A} \widehat{\otimes}_{\mathcal{B}} \widehat{\widehat{\mathbb{V}}}$$

Here by  $\widehat{\otimes}$  we mean the completed tensor product. Namely,

$$\mathcal{A} \widehat{\otimes}_{\mathcal{B}} \widehat{\widehat{\mathbb{V}}} = \varprojlim_{N \rightarrow \infty} \mathcal{A} \otimes_{\mathcal{B}} \widehat{\widehat{\mathbb{V}}} / \exp\left(\frac{N}{i\hbar}\right) \mathcal{A}_{\Lambda} \otimes_{\mathcal{B}} \widehat{\widehat{\mathbb{V}}}$$

In 13 we interpret  $\widehat{\mathcal{V}}$  as an algebraic version of the metaplectic representation (Proposition 13.8).

**9.2.2. The sheaf of algebras  $\mathcal{B}_L$  and the sheaf of modules  $\widehat{\widehat{\mathbb{V}}}_L$ .** Let  $L$  be a Lagrangian submanifold of  $M$ . Recall that we assume the existence of an  $\mathrm{Sp}^4$  structure on  $M$ . Consider the restriction to  $L$  of the  $\mathrm{Sp}^4(n)$ -valued cocycle  $\tilde{g}_{jk}$  as in 7.2.1 or in 7.2.2 (it does not matter which one of them because  $\omega|_L = 0$ ). Consider the cohomologous  $\mathrm{MPar}(n)$ -valued cocycle  $\tilde{p}_{jk}$  as in (12.1.4).

The group  $\mathrm{MPar}(n, \mathbb{R})$  (cf. 12.1) acts on  $\mathcal{B}$  by automorphisms. It also acts on  $\widehat{\widehat{\mathbb{V}}}_{\mathbb{K}}$  compatibly. Let  $\mathcal{B}_L$  be the bundle of algebras and  $\widehat{\widehat{\mathbb{V}}}_L$  the bundle of modules on  $L$  associated to these actions and to the principal  $\mathrm{MPar}$ -bundle defined by  $p_{jk}$ . Note that the Lie algebra  $\tilde{\mathfrak{g}}$  (3.1.4) acts by derivations on  $\mathcal{B}$  and on  $\widehat{\widehat{\mathbb{V}}}_{\mathbb{K}}$ . Therefore any given Fedosov connection defines a connection on  $\mathcal{B}_L$  and on  $\widehat{\widehat{\mathbb{V}}}_L$ . If the curvature of this connection is  $\frac{1}{i\hbar}\omega$  then the connection on  $\widehat{\widehat{\mathbb{V}}}_L$  is flat. We denote these connections by  $\nabla_{\mathcal{B}}$  and  $\nabla_{\mathbb{V}}$ .

**Definition 9.6.** By  $\mathcal{B}_L^\bullet$ , resp. by  $\widehat{\widehat{\mathbb{V}}}_L^\bullet$ , we denote the differential graded algebra of  $\mathcal{B}_L$ -valued  $\mathbb{K}$ -forms with differential  $\nabla_{\mathcal{B}}$ , resp. the differential graded module of  $\widehat{\widehat{\mathbb{V}}}_L$ -valued  $\mathbb{K}$ -forms with differential  $\nabla_{\mathbb{V}}$ .

**9.2.3. The Lie groupoid  $\mathbf{P}_L$ .** (12.1.4) Construct the groupoid  $\mathbf{P}_L$  as the groupoid of the (twisted in general, but not in this case) bundle defined by this cocycle as in 14.2. We have a short exact sequence of groups

$$(9.2.6) \quad 1 \rightarrow \mathrm{MPar}(n, \mathbb{R}) \rightarrow (\mathbf{P}_L)_{x,x} \rightarrow \pi_1(L, x) \rightarrow 1$$

for every point  $x$  of  $L$ .

**Definition 9.7.** Let  $\widehat{\widehat{\mathbb{V}}}_L^\bullet$  be the  $\mathcal{B}_L^\bullet$ -module with the compatible action of  $\mathbf{P}_L$  and the compatible connection  $\nabla_{\mathbb{V}}$  as in Definition 9.6. The oscillatory module  $\mathcal{V}_L^\bullet$  is the  $\mathcal{A}_M^\bullet$ -module with a compatible action of  $\tilde{\mathbf{G}}_M$  and a compatible connection induced from  $\widehat{\widehat{\mathbb{V}}}_L^\bullet$  as in Definition 9.2.

### 9.3. The filtration on $\mathbb{R}\mathrm{HOM}$ .

**Conjecture 9.8.** Assume that  $L_0$  and  $L_1$  are *graded* Lagrangian submanifolds as in [44], *i.e.* their Maslov classes with values in  $H^1(L_j, \mathbb{Z}/4\mathbb{Z})$  are trivial. Then  $\mathbb{R}\mathrm{HOM}(\mathcal{V}_{L_0}^\bullet, \mathcal{V}_{L_1}^\bullet)$  is a filtered  $A_\infty$  local system.

### 9.4. The case of $\mathbb{R}^{2n}$ .

**9.4.1. The groupoid  $\tilde{\mathbf{G}}_M$ .** Sections of  $\tilde{\mathbf{G}}_M$  are in bijection with smooth functions  $g(x_1, \xi_1; x_2, \xi_2)$  on  $M \times M$  with values in  $\mathrm{Sp}^4(n)$ . We will denote a section corresponding to  $g$  by a formal symbol

$$(9.4.1) \quad \sigma(x_1, \xi_1; x_2, \xi_2) = \exp\left(\frac{1}{i\hbar}(\xi_2 - \xi_1)\hat{x} + (x_1 - x_2)\hat{\xi}\right)g(x_1, \xi_1; x_2, \xi_2)$$

The composition consists of formal multiplication of exponentials and multiplication of elements of  $\mathrm{Sp}^4(2n)$ .

**9.4.2. The flat connection up to inner derivations on  $\mathcal{A}_M$  compatible with the action of  $\tilde{\mathbf{G}}_M$ .** For a section  $\sigma$  as in (9.4.1),

$$\begin{aligned} -\alpha(\sigma) &= \nabla_{\tilde{\mathbf{G}}} \sigma \cdot \sigma^{-1} = d_{\text{DR}} g \cdot g^{-1} + \frac{1}{i\hbar} (\xi_2 dx_2 - \xi_1 dx_1) + \\ &\quad \left( -\frac{\hat{\xi}_1}{i\hbar} dx_1 + \frac{\hat{x}_1}{i\hbar} d\xi_1 \right) - \text{Ad}_g \left( -\frac{\hat{\xi}_2}{i\hbar} dx_2 + \frac{\hat{x}_2}{i\hbar} d\xi_2 \right) \end{aligned}$$

**9.4.3. The sheaf  $\mathcal{V}_f^\bullet$ .** Denote by  $\mathcal{V}_f^\bullet$  the oscillatory module corresponding to the Lagrangian submanifold  $\text{graph}(df)$ . One has

$$(9.4.2) \quad \mathcal{V}_f^\bullet = \hat{\mathcal{V}}_M^\bullet = \Omega_{\mathbb{K}, M}^\bullet(\hat{\mathcal{V}}).$$

In other words, local sections of  $\mathcal{V}_f^\bullet$  are  $\mathcal{V}$ -valued  $\mathbb{K}$ -forms on  $M$  (cf. Definition 9.5).

*Remark 9.9.* Sections of (9.4.2) are identified with elements of  $\mathcal{V}_f$  of the form  $\sigma((x, \xi); (x, x - f'(x)))w$  (cf. (9.4.1)) where  $w$  are sections of  $\mathbb{V}_L$  (cf. (9.2)).

**9.4.4. The connection on  $\mathcal{V}_f$ .**

$$(9.4.3) \quad \nabla_{\mathcal{V}} = -\frac{\xi - f'(x)}{i\hbar} dx + \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial \hat{x}} - \frac{1}{i\hbar} f''(x) \hat{x} \right) dx + \left( \frac{\partial}{\partial \xi} + \frac{1}{i\hbar} \hat{x} \right) d\xi$$

**9.4.5. The action of  $\hat{\mathbb{A}}$  on  $\mathcal{V}_f$ .** The formal variables act as follows:  $\hat{x}$  by multiplication, and  $\hat{\xi}$  by  $i\hbar \frac{\partial}{\partial \hat{x}} + f'(x + \hat{x}) - f'(x)$ .

**9.4.6. The action of  $\tilde{\mathbf{G}}_M$  on  $\mathcal{V}_f$ .** A section  $\sigma$  as in (9.4.1) acts by

$$(9.4.4) \quad \exp\left(-\frac{1}{i\hbar} (f(x_1 + \hat{x}) - f'(x_1)\hat{x} - f(x_2 + \hat{x}) - f'(x_2)\hat{x})\right) g(x_1, \xi_1; x_2, \xi_2)$$

## 10. THE COMPLEX COMPUTING $\mathbb{R}\text{Hom}(\mathcal{V}_0^\bullet, \mathcal{V}^\bullet)$

**10.1. The statement of the result.** Let  $M = \mathbb{R}^{2n}$ . Given an oscillatory module  $\mathcal{V}^\bullet$  on  $M$ , construct the following complex. Note first that the group  $\text{MPar}(n, \mathbb{R})$  acts on the linear span of  $d\hat{x}_1, \dots, d\hat{x}_n$  through the projection  $\text{MPar}(n) \rightarrow \text{ML}(n)$ . Introduce the vector space

$$(10.1.1) \quad \wedge (d\hat{x}_1, \dots, d\hat{x}_n) d^{-\frac{1}{2}} \hat{x}$$

where

$$d^{\frac{1}{2}} \hat{x} = (d\hat{x}_1 \dots d\hat{x}_n)^{-\frac{1}{2}}$$

is a formal element on which a pair  $(g, u)$  in  $\text{MPar}$  acts via multiplication by  $u$ . Consider the space

$$(10.1.2) \quad \wedge (d\hat{x}_1, \dots, d\hat{x}_n) d^{-\frac{1}{2}} \hat{x} \otimes \mathcal{V}^\bullet$$

with the following structures.

10.1.1. **The differential.** Define the differential on (10.1.2) as

$$\tilde{\nabla}_{\mathcal{V}} = \frac{1}{i\hbar}(\xi dx + \hat{\xi} d\hat{x} - \hat{x} d\xi) + \nabla_{\mathcal{V}}$$

One checks that  $\tilde{\nabla}^2 = 0$ . In fact,

$$\begin{aligned} \frac{1}{i\hbar} \nabla_{\mathcal{V}}(\xi dx + \hat{\xi} d\hat{x} - \hat{x} d\xi) + \frac{1}{(i\hbar)^2}(\xi dx + \hat{\xi} d\hat{x} - \hat{x} d\xi)^2 = \\ \frac{1}{i\hbar}(-d\xi d\hat{x} + d\xi dx + dx d\xi - d\hat{x} d\xi) = 0 \end{aligned}$$

10.1.2. **The action of MPar.** Denote by  $\text{MPar}(n, \mathbb{R})_M$  the sheaf of smooth sections of the associated (in our case trivial) bundle of groups with fiber  $\text{MPar}(n)$ .

There is an obvious action of  $\text{MPar}(n, \mathbb{R})_M$  on (10.1.2) but we have to modify it to make it commute with the differential. Put

$$(10.1.3) \quad \mathbf{R}_h = h + \frac{1}{i\hbar}[\iota_{d\hat{x}} dx, h]$$

Here  $[\cdot, \cdot]$  stands for the commutator of operators on (10.1.2);

$$\iota_{d\hat{x}} dx = \sum_{j=1}^n \iota_{d\hat{x}_j} dx_j;$$

and  $\iota_{d\hat{x}_j}$  is the graded derivation of  $\wedge(d\hat{x}_1, \dots, d\hat{x}_n)$  that sends  $d\hat{x}_j$  to one and  $d\hat{x}_k$  to zero for  $k \neq j$ . One checks immediately that

$$(10.1.4) \quad \mathbf{R}_{h_1} \mathbf{R}_{h_2} = \mathbf{R}_{h_1 h_2}$$

**Lemma 10.1.**

$$(10.1.5) \quad \tilde{\nabla}_{\mathcal{V}} \mathbf{R}_h = \mathbf{R}_h \tilde{\nabla}_{\mathcal{V}}$$

*Proof.* For a local section  $h$  of  $\text{MPar}(n)_M$ , define  $\alpha(h) \in \Omega_M^1(\mathcal{A}_M)$  by

$$(10.1.6) \quad \alpha(h) = -dh \cdot h^{-1} + A_{-1} - \text{Ad}_h(A_{-1})$$

where  $A_{-1} = \frac{1}{i\hbar}(-\hat{\xi} dx + \hat{x} d\xi)$ . Note that

$$(10.1.7) \quad \nabla_{\mathcal{V}}(\mathbf{R}_h v) = -\alpha(h) \mathbf{R}_h v + \mathbf{R}_h \nabla_{\mathcal{V}} v;$$

$$(10.1.8) \quad -\frac{1}{i\hbar}(\xi dx + \hat{\xi} d\hat{x} - \hat{x} d\xi)(\mathbf{R}_h v) = -\alpha(h) \mathbf{R}_h v - \mathbf{R}_h \frac{1}{i\hbar}(\xi dx + \hat{\xi} d\hat{x} - \hat{x} d\xi)v$$

The first equation is equivalent to the fact that  $\mathcal{V}^\bullet$  is a differential graded  $\mathcal{A}_M^\bullet$ -module. The second is checked by a direct computation:

$$\begin{aligned} \frac{1}{i\hbar}[\hat{\xi} dx + \xi dx - \hat{x} d\xi, h] &= -\frac{1}{i\hbar}[\hat{x} d\xi, h]; \\ \frac{1}{i\hbar}[\hat{\xi} dx + \xi dx - \hat{x} d\xi, [\iota_{d\hat{x}} dx, h]] &= \frac{1}{i\hbar}[[\hat{\xi} dx + \xi dx - \hat{x} d\xi, \iota_{d\hat{x}}] dx, h] + \\ &[\iota_{d\hat{x}} dx, \frac{1}{i\hbar}[\hat{\xi} dx + \xi dx - \hat{x} d\xi, h]] = \frac{1}{i\hbar}[\hat{\xi} dx, h] \end{aligned}$$

(the second summand vanishes). Therefore

$$\frac{1}{i\hbar}[\widehat{\xi}dx + \xi dx - \widehat{x}d\xi, \mathbf{R}_h] = \frac{1}{i\hbar}[\widehat{\xi}dx - \widehat{x}d\xi, h] = -[\nabla_{\mathcal{V}}, h].$$

Lemma 10.1.5 immediately follows.  $\square$

**Proposition 10.2.** *The standard complex computing group cohomology*

$$C^\bullet(\text{MPar}(n)_M, \wedge(d\widehat{x}_1, \dots, d\widehat{x}_n)d^{-\frac{1}{2}}\widehat{x} \otimes \mathcal{V}^\bullet)$$

*is quasi-isomorphic to the complex  $\mathcal{C}^\bullet(\mathcal{V}_0^\bullet, \mathcal{A}_M^\bullet, \mathcal{V}^\bullet)$ .*

More precisely,

$$\begin{aligned} C^\bullet &= \oplus_{m=0}^\infty \text{Hom}((\text{MPar}(n)_M)^m, \wedge(d\widehat{x}_1, \dots, d\widehat{x}_n)d^{-\frac{1}{2}}\widehat{x} \otimes \mathcal{V}^\bullet); \\ (\delta D)(h_1, \dots, h_{m+1}) &= (-1)^m \widetilde{\nabla}_{\mathcal{V}} D(h_1, \dots, h_{m+1}) + \mathbf{R}_{h_1} D(h_2, \dots, h_{m+1}) + \\ &+ \sum_{j=1}^m (-1)^j D(h_1, \dots, h_j h_{j+1}, \dots, h_{m+1}) + (-1)^{m+1} D(h_1, \dots, h_m); \end{aligned}$$

The following section 10.2 is devoted to the proof of Proposition 10.2.

## 10.2. The resolution of $\mathcal{V}_0$ and the computation of $\mathbb{R}\text{Hom}(\mathcal{V}_0, \mathcal{V})$ .

**10.2.1. A resolution of  $\mathcal{V}_0$ .** As above, let  $M = \mathbb{R}^{2n}$ . First construct a resolution  $\mathbb{P}^\bullet$  that is only free over  $\widehat{\mathbb{A}}_M$ , not over  $\mathcal{A}_M$ . This resolution is a free module over

$$(10.2.1) \quad \widehat{\widehat{\mathbb{A}}^\bullet}_M = \Omega_M^\bullet(\widehat{\mathbb{A}}_M)$$

with the space of generators

$$\wedge(e_1, \dots, e_n)v_0; |v_0| = 0; |e_j| = -1$$

with the differential  $\nabla_{\mathbb{P}}$  defined by the following properties:

$$(10.2.2) \quad \nabla_{\mathbb{P}}v_0 = \frac{1}{i\hbar}(-\xi dx + \widehat{x}d\xi)v_0; \nabla e_j = \widehat{\xi}_j;$$

$$\nabla_{\mathbb{P}}(av) = \nabla_{\mathcal{A}}a \cdot v + (-1)^{|a|}a\nabla_{\mathbb{P}}v$$

for any  $a$  in  $\widehat{\widehat{\mathbb{A}}^\bullet}_M$  and any  $v$  in  $\mathbb{P}$ ; and

$$\nabla_{\mathbb{P}}(\beta v_0) = \nabla_{\mathbb{P}}\beta \cdot v_0 + (-1)^{|\beta|}\beta\nabla_{\mathbb{P}}v_0$$

for any  $\beta$  in  $\wedge(e_1, \dots, e_n)$ . A simple computation shows that  $\nabla^2 = 0$ .

Next we construct a  $\mathcal{B}_M^\bullet$ -free resolution of the  $\mathcal{B}_M^\bullet$ -module  $\widehat{\mathcal{V}}_M^\bullet$ . Here, as always,  $\mathcal{B}_M^\bullet$  stands for forms with coefficients in the (trivial) bundle of algebras associated to  $\mathcal{B}$ , and  $\widehat{\mathcal{V}}_M^\bullet$  stands for forms with coefficients in the bundle of modules associated to  $\widehat{\mathcal{V}}$ , cf. Definition 9.5. We first observe that  $\mathbb{P}^\bullet$  is in fact a  $\mathcal{A}^\bullet$ -module, though not free. Indeed, to define an  $\mathcal{B}_M^\bullet$ -action, we have to define an  $\text{MPar}_M$ -action compatible with the action of the smaller algebra and with the differential. We are going to do this next.

**10.2.2. The action of  $\text{MPar}(n)_M$ .** The action of  $\text{MPar}(n)_M$  extends from  $\widehat{\mathcal{V}}_M^\bullet$  to  $\mathbb{P}^\bullet$  because of the following. The group  $\text{MPar}$  also acts on  $\wedge(e_1, \dots, e_n)$ . The latter action is induced by the linear action on  $\mathbb{R}^n$  which in our context is the easiest to describe as follows: identify  $e_j$  with  $\widehat{\xi}_j$  and therefore  $\mathbb{R}^n$  with the linear span of  $\widehat{\xi}_j$  in  $\widehat{\mathbb{A}}$ . The action of  $\text{MPar}$  through the composition  $\text{MPar} \rightarrow \text{GL} \rightarrow \text{Sp}$  on  $\widehat{\mathbb{A}}$  leaves this subspace invariant. This is the action that we mean.

Recall again that an element of  $\text{MPar}(n)$  may be represented by a pair

$$\left( \begin{bmatrix} b & a \\ 0 & b^{t-1} \end{bmatrix}, u \right); \det(b) = u^2.$$

This element sends  $v_0$  to  $u^{-1}v_0$ . Combined with the above, we get an action of  $\text{MPar}(n)$  on  $\wedge(e_1, \dots, e_n)v_0$ .

Unfortunately, this action does not make  $\mathbb{P}^\bullet$  a differential graded  $\mathcal{B}_M$ -module. To achieve that, we have to change the action as follows:

$$(10.2.3) \quad \mathbf{R}_h = h + \left[ \frac{1}{i\hbar} \text{edx}, h \right]$$

Here  $\text{edx} = \sum_j e_j dx_j$ . The commutator is just the commutator of operators on  $\mathbb{P}^\bullet$ . This action, unlike the previous one, makes  $\mathbb{P}^\bullet$  a differential graded  $\mathbb{P}^\bullet$ -module, which is equivalent to the following.

One has

$$(10.2.4) \quad \nabla_{\mathbb{P}}(\mathbf{R}_h v) = -\alpha(h)\mathbf{R}_h v + \mathbf{R}_h \nabla_{\mathbb{P}} v$$

**10.2.3. The resolution  $\mathcal{P}^\bullet$ .** Now define

$$(10.2.5) \quad \mathcal{P}^\bullet = \mathcal{B}_{-\bullet}(\text{MPar}(n)_M, \mathbb{P}^\bullet) = \oplus_{m=0}^{\infty} \mathbb{C}[\text{MPar}(n)_M]^{\otimes m} \widehat{\otimes} \mathbb{P}^\bullet$$

The action of  $\mathcal{B}_M$  on  $\mathcal{P}^\bullet$  is given by

$$h((h_1, \dots, h_m) \otimes v) = (hh_1, \dots, h_m) \otimes \mathbf{R}_h v$$

(cf. (10.2.3));

$$a((h_1, \dots, h_m) \otimes v) = (h_1, \dots, h_m) \otimes av$$

for  $h$  in  $\text{MPar}(n)_M$  and  $a$  in  $\widehat{\mathbb{A}}_M$ .

This is the standard bar resolution of the  $\text{MPar}$ -module  $\mathbb{P}^\bullet$ . More precisely, the differential is given by

$$\nabla_{\mathcal{P}} = \nabla_{\mathcal{P}}^{(0)} + \nabla_{\mathcal{P}}^{(1)}$$

$$(10.2.6) \quad \nabla^{(0)}((h_1, \dots, h_m) \otimes v) = (-1)^m (h_1, \dots, h_m) \otimes \nabla_{\mathbb{P}} v$$

$$(10.2.7) \quad \begin{aligned} \nabla^{(1)}((h_1, \dots, h_m) \otimes v) &= \sum_{j=1}^{m-1} (-1)^j (h_1, \dots, h_j h_{j+1}, \dots, h_m) \otimes v \\ &\quad + (-1)^m (h_1, \dots, h_{m-1}) \otimes v \end{aligned}$$

Finally, put

$$(10.2.8) \quad \mathcal{R}^\bullet = \mathcal{A}_M^\bullet \widehat{\otimes}_{\mathcal{B}_M^\bullet} \mathcal{P}^\bullet$$

10.3. **The complex**  $\text{Hom}(\mathcal{R}^\bullet, \mathcal{V}^\bullet)$ . The complex

$$(10.3.1) \quad \text{Hom}_{\mathcal{A}^\bullet}(\mathcal{R}^\bullet, \mathcal{V}^\bullet)$$

is now straightforward to compute for any oscillatory module  $\mathcal{V}$  on  $\mathbb{R}^{2n}$ . It is the complex of cochains of the group  $\text{MPar}(n)_M$  with coefficients in the module  $\wedge(e_1^*, \dots, e_n^*)v_0^* \otimes \mathcal{V}$ ,

$$(10.3.2) \quad \text{Hom}_{\mathcal{A}^\bullet}(\mathcal{R}^\bullet, \mathcal{V}^\bullet) \xrightarrow{\sim} C^\bullet(\text{MPar}(n)_M, \wedge(e_1^*, \dots, e_n^*)v_0^* \otimes \mathcal{V})$$

Here  $|e_j^*| = 1$ ;  $|v_0^*| = 0$ ; the action of  $\text{MPar}$  on  $\wedge(e_1^*, \dots, e_n^*)v_0^*$  is dual to the one from 10.2.2. It is straightforward that this complex is identical to the one in Proposition 10.2.

10.3.1. **The case**  $\mathcal{V} = \mathcal{V}_f$ . Now we are able to compute  $\mathbb{R} \text{Hom}_{\mathcal{A}_M^\bullet}(\mathcal{V}_0^\bullet, \mathcal{V}_f^\bullet)$ . Recall (Definition 9.2.3)

$$\widehat{\mathbb{V}}_{\mathbb{K}} = \left\{ \sum_{k=0}^{\infty} e^{\frac{1}{i\hbar} c_k} v_k \mid \in \widehat{\mathbb{V}}; c_k \in \mathbb{R}; c_k \rightarrow \infty \right\}$$

Here we view this space with the following action of  $\text{MPar}(n, \mathbb{R})$  :

$$\begin{aligned} \begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix} &\mapsto S_b, (S_b f)(x) = f(b^{-1}x); \\ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} &\mapsto \exp\left(-\frac{i\hbar}{2} a \frac{\partial^2}{\partial \widehat{x}}\right) \end{aligned}$$

Now define the  $\text{MPar}(n)$ -module

$$(10.3.3) \quad \Omega_{\mathbb{K}}^{\bullet, \bullet} = \wedge(d\widehat{x}_1, \dots, d\widehat{x}_n) \otimes \mathbb{C}[\text{Sp}^4(n)] \otimes_{\text{MPar}(n)} \widehat{\mathbb{V}}_{\mathbb{K}}$$

and the  $\text{MPar}(n)_M$ -module of  $\Omega_{\mathbb{K}}^{\bullet, \bullet}$  forms with coefficients in (10.3.3).

*Remark 10.3.* Intuitively,  $\Omega_{\mathbb{K}}^{\bullet, \bullet}$  is the space of expressions

$$(10.3.4) \quad \sum_{J, K, j} \exp\left(\frac{1}{i\hbar} \varphi_{j, J, K}(x, \xi, \widehat{x})\right) a_{j, J, K}(x, \xi, \widehat{x}) dx_J d\widehat{x}_K$$

where linear term of  $\varphi_{j, J, K}(x, \xi, \widehat{x})$  with respect to  $\widehat{x}$  is zero, and its quadratic term may be infinite; more precisely, it is allowed to be not just a quadratic form but a point of the Lagrangian Grassmannian.

The differential on  $\wedge(d\widehat{x}_1, \dots, d\widehat{x}_n) \otimes \widehat{\mathcal{V}}_{\mathbb{K}}^\bullet$  is

$$(10.3.5) \quad d_f = \frac{\partial}{\partial \xi} d\xi + \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial \widehat{x}}\right) dx + \frac{\partial}{\partial \widehat{x}} d\widehat{x} + \frac{1}{i\hbar} (f'(x + \widehat{x}) - f'(x)) d\widehat{x} + \frac{1}{i\hbar} (f'(x) - f''(x)\widehat{x}) dx$$

One has

$$(10.3.6) \quad d_f = \left(\exp\left(-\frac{1}{i\hbar} (f(x + \widehat{x}) - f'(x)\widehat{x})\right) d_0 \left(\exp\left(\frac{1}{i\hbar} (f(x + \widehat{x}) - f'(x)\widehat{x})\right)\right)\right)$$



**Proposition 10.4.** *The standard complex  $\mathcal{C}^{\bullet}_{\mathcal{A}_M}(\mathcal{V}_0^{\bullet}, \mathcal{V}_f^{\bullet})$  is quasi-isomorphic to the complex*

$$(10.3.7) \quad C^{\bullet}(\text{MPar}(n)_M, \Omega_{\mathbb{K}}^{\bullet, \bullet}).$$

### 10.3.2. A stationary phase statement.

**Lemma 10.5.** *For any positive integer  $p$ , consider  $\mathbb{R}^p$  viewed as a discrete group. One has*

$$H^{\bullet}(\mathbb{R}^p, \mathbb{C}[\mathbb{R}^p]) = 0.$$

*Proof.* One has  $\mathbb{R}^p \xrightarrow{\sim} \bigoplus \mathbb{Q}$ . Therefore  $\mathbb{R}^p \xrightarrow{\sim} \mathbb{Q} \oplus \mathbb{R}^p$ . By Künneth formula,

$$H^{\bullet}(\mathbb{R}^p, \mathbb{C}[\mathbb{R}^p]) \xrightarrow{\sim} H^{\bullet}(\mathbb{Q}, \mathbb{Q}) \otimes H^{\bullet}(\mathbb{R}^p, \mathbb{C}[\mathbb{R}^p]).$$

But  $H^0(\mathbb{Q}, \mathbb{Q}) = 0$ . If  $k$  is the minimal integer such that  $H^k(\mathbb{R}^p, \mathbb{C}[\mathbb{R}^p]) \neq 0$ , Künneth formula tells that  $H^k = 0$ , whence the contradiction.  $\square$

**Corollary 10.6.** *Let  $\Omega$  be an orbit of  $\text{MPar}(n, \mathbb{R})$  in the Lagrangian Grassmannian  $\Lambda(n)$  that consists of more than one point. Then*

$$H^{\bullet}(\text{MPar}(n), \mathbb{C}[\Omega]) = 0.$$

*Proof.* Let  $N$  be the subgroup of  $\text{MPar}(n, \mathbb{R})$  consisting of pairs

$$\left( \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, 1 \right)$$

(in other words,  $N = \text{Ker}(\text{MPar}(n) \rightarrow \text{ML}(n))$ ). Choose a point in  $\Omega$ . Denote its stabilizer by  $Z$ . Then  $Z$  is a real vector subspace of  $N$ . Let  $W$  be a complementary subspace to  $Z$ . Consider the Lyndon spectral sequence

$$E_2^{pq} = H^p(N/Z, H^q(Z, \mathbb{C}[\Omega])) \implies H^{p+q}(N, \mathbb{C}[\Omega]).$$

But  $\Omega \xrightarrow{\sim} Z$  as a  $Z$ -set, so  $H^{\bullet}(N, \mathbb{C}[\Omega]) = 0$  by Lemma 10.5. Now consider the Lyndon spectral sequence

$$E_2^{pq} = H^p(\text{ML}(n), H^q(N, \mathbb{C}[\Omega])) \implies H^{p+q}(\text{MPar}(n), \mathbb{C}[\Omega]).$$

The statement follows.  $\square$

### 10.4. The computation of $\mathbb{R}\text{HOM}(\mathcal{V}_0, \mathcal{V}_f)$ .

$$(10.4.1) \quad \mathcal{S}^{\bullet} = C^{\bullet}(\text{MPar}(n), \mathbb{K})$$

**Theorem 10.7.**

$$\mathbb{R}\text{HOM}^{\bullet}(\mathcal{V}_0^{\bullet}, \mathcal{V}_f^{\bullet}) \xrightarrow{\sim} \mathcal{S}^{\bullet}$$

*with the action of a path from  $(x_1, \xi_1)$  to  $(x_2, \xi_2)$  given by multiplication by  $\exp(\frac{1}{i\hbar}(f(x_1) - f(x_2)))$ .*

*Proof.* First one checks that all the structures for  $\mathcal{V}_0^{\bullet}$  and  $\mathcal{V}_f^{\bullet}$  are conjugate by multiplication by  $\exp(\frac{1}{i\hbar}(f(x + \hat{x}) - f'(x)\hat{x}))$ . So we can reduce the statement to the case  $f = 0$ . The cohomology in question is computed by the complex

$$(10.4.2) \quad C^{\bullet}(\pi_1(M), C^{\bullet}(\text{MPar}(n)_M, \Omega^{\bullet, \bullet})).$$

First compute the cohomology of  $\pi_1(M)$ . An argument identical to the one in Introduction (starting before (1.1.6)) shows that this cohomology is isomorphic to

$$(10.4.3) \quad H^\bullet(\text{MPar}(n), \mathbb{C}[\text{Sp}^4] \hat{\otimes}_{\mathbb{C}[\text{MPar}(n)]} \hat{\widehat{\mathbb{V}}}_{\mathbb{K}})$$

(In other words, all dependence on  $x, \xi$ , and  $dx, d\xi$  is eliminated). Now, by Corollary 10.6, all contributions from all Lagrangian submanifolds other than  $L_0 = \{\xi = 0\}$  are also eliminated. Our cohomology is therefore computed by the complex

$$(10.4.4) \quad C^\bullet(\text{MPar}(n), \wedge(d\hat{x}_1, \dots, d\hat{x}_n) \otimes \hat{\widehat{\mathbb{V}}}_{\mathbb{K}})$$

of group cochains of  $\text{MPar}(n)$  with coefficients in the complex  $\wedge(d\hat{x}_1, \dots, d\hat{x}_n) \otimes \hat{\widehat{\mathbb{V}}}_{\mathbb{K}}$  of formal forms in  $\hat{x}$  with the differential  $\frac{\partial}{\partial \hat{x}} d\hat{x}$ .  $\square$

**10.5. The case of sheaves.** Here we compare the computation above to the analogous computation for the microlocal category of sheaves as in 1.7.

**Proposition 10.8.** *Let  $f$  and  $g$  be two  $C^\infty$  functions on  $\mathbb{R}^n$ . For a bounded contractible open subset of  $\mathbb{R}^n$ , the module of horizontal sections of the local system  $\mathbb{R}\text{HOM}(\mathcal{V}_g^\bullet, \mathcal{V}_f^\bullet)$  on  $U$  is a free  $\mathcal{S}^\bullet$ -module with one generator  $J(f, g)$  lying in  $\text{Filt}^{-\inf_U(f-g)}$ . The composition is as follows:*

$$J(f, g)J(g, h) = \exp\left(\frac{1}{i\hbar}c(f, g, h)\right)J(f, h)$$

where

$$c(f, g, h) = \inf_U(f - h) - \inf_U(f - g) - \inf_U(g - h)$$

*Proof.* It is easy to see that

$$\mathbb{R}\text{HOM}(\mathcal{V}_g^\bullet, \mathcal{V}_f^\bullet) \xrightarrow{\sim} \mathbb{R}\text{HOM}(\mathcal{V}_0^\bullet, \mathcal{V}_{f-g}^\bullet)$$

Put

$$(10.5.1) \quad J(f, g) = \exp\left(\frac{1}{i\hbar}((f - g)(x + \hat{x}) - (f - g)'(x)\hat{x} - \inf_U(f - g))\right)$$

The statement follows from Theorem 10.7.  $\square$

Compare this to the following result of Tamarkin. Recall the definitions from 1.7.1. Put

$$\mathbb{K}_{\mathbb{Z}} = \left\{ \sum_{k=0}^{\infty} a_k e^{-\frac{c_k}{i\hbar}} \right\}$$

where  $a_k \in \mathbb{Z}$ ,  $c_k \in \mathbb{R}$ , and  $c_k \rightarrow \infty$ . For any two objects  $\mathcal{F}$  and  $\mathcal{G}$  of  $D(T^*\mathbb{R}^n)$ , let  $\text{HOM}_{\mathbb{K}}(\mathcal{F}, \mathcal{G}) = \mathbb{K}_{\mathbb{Z}} \otimes_{\Lambda_{\mathbb{Z}}} \text{HOM}(\mathcal{F}, \mathcal{G})$ . Let  $\text{Filt}^c \text{HOM}_{\mathbb{K}} = e^{\frac{c}{i\hbar}} \text{HOM}$ .

**Proposition 10.9.** *Let  $f$  and  $g$  be two  $C^\infty$  functions on  $\mathbb{R}^n$ . For a bounded contractible open subset  $U$  of  $\mathbb{R}^n$ , consider the objects  $\mathcal{F}_f$  and  $\mathcal{F}_g$  of  $D(T^*U)$  as in 1.7.1. The complex  $\mathrm{HOM}_{\mathbb{K}}(\mathcal{F}_g, \mathcal{F}_f)$  is quasi-isomorphic to a free  $\mathbb{K}_{\mathbb{Z}}$ -module with one generator  $J(f, g)$  lying in  $\mathrm{Filt}^{-\inf_U(f-g)}$ . The composition satisfies the same formulas as in Proposition 10.8.*

*Proof.* Recall that  $\mathcal{F}_f = \mathbb{Z}_{t+f \geq 0}$ . It is immediate that

$$(10.5.2) \quad \mathrm{HOM}_{\mathbb{K}}(\mathcal{F}_g, \mathcal{F}_f) \xrightarrow{\sim} \mathrm{HOM}_{\mathbb{K}}(\mathcal{F}_0, \mathcal{F}_{f-g})$$

Let  $J(f, g)$  be the morphism  $\mathbb{Z}_{t \geq 0} \rightarrow \mathbb{Z}_{t+f-g-\inf_U(f-g) \geq 0}$  which is the restriction to the subset  $\{t+f-g-\inf_U(f-g) \geq 0\} \subset \{t \geq 0\}$ . It is clear that the right hand side of (10.5.2) is the free  $\mathbb{K}_{\mathbb{Z}}$ -module generated by  $J(f, g)$ , that  $J(f, g)$  is in  $\mathrm{Filt}^{-\inf_U(f-g)}$ , and that the composition is as in Proposition 10.8.  $\square$

10.5.1. **Matrix units.** Now put

$$(10.5.3) \quad \mathbf{E}_{f,g} = \exp\left(\frac{1}{i\hbar} \inf_U(f-g)\right) J(f, g) \in \mathrm{HOM}_{\mathbb{K}}(\mathcal{F}, \mathcal{G})$$

in  $D(T^*U)$ . Then

$$(10.5.4) \quad \mathbf{E}_{f,g} \mathbf{E}_{g,h} = \mathbf{E}_{f,h}$$

## 11. $\mathbb{R}\mathrm{Hom}$ AND THETA FUNCTIONS

11.1. **Modules associated to the Lagrangian submanifold  $\xi = mx$ .** In this section,  $M = \mathbb{T}^2$  and  $\widetilde{M} = \mathbb{R}^2$  with the standard symplectic form  $\omega = d\xi dx$ .

11.1.1. **The groupoid  $\widetilde{\mathbf{G}}_M$ .** Local sections of  $\widetilde{\mathbf{G}}_M$  are in bijection with smooth local functions  $g(x_1, \xi_1; x_2, \xi_2)$  on  $\widetilde{M} \times \widetilde{M}$  with values in  $\mathrm{Sp}^4(2)$ . As in 9.4.1, we denote a section corresponding to  $g$  by a formal symbol

$$(11.1.1) \quad \sigma(x_1, \xi_1; x_2, \xi_2) = \exp\left(\frac{1}{i\hbar}(\xi_2 - \xi_1)\widehat{x} + (x_1 - x_2)\widehat{\xi}\right) g(x_1, \xi_1; x_2, \xi_2)$$

These sections satisfy

$$(11.1.2) \quad \sigma(x_1, \xi_1; x_2, \xi_2) = \exp\left(\frac{1}{i\hbar}(x_1 - x_2)\right) \sigma(x_1, \xi_1 + 1; x_2, \xi_2 + 1);$$

$$(11.1.3) \quad \sigma(x_1, \xi_1; x_2, \xi_2) = \sigma(x_1 + 1, \xi_1; x_2 + 1, \xi_2).$$

As in 9.4.1, the composition consists of formal multiplication of exponentials and multiplication of elements of  $\mathrm{Sp}^4(2)$ .

The flat connection up to inner derivations on  $\widetilde{\mathbf{G}}_M$  is given exactly as in 9.4.4: for a section  $\sigma$  as in (9.4.1),

$$-\alpha(\sigma) = \nabla_{\widetilde{\mathbf{G}}} \sigma \cdot \sigma^{-1} = d_{\mathrm{DR}} g \cdot g^{-1} + \frac{1}{i\hbar} (\xi_2 dx_2 - \xi_1 dx_1) +$$

$$(-\frac{\widehat{\xi}_1}{i\hbar}dx_1 + \frac{\widehat{x}_1}{i\hbar}d\xi_1) - \text{Ad}_g(-\frac{\widehat{\xi}_2}{i\hbar}dx_2 + \frac{\widehat{x}_2}{i\hbar}d\xi_2)$$

**11.1.2. The sheaf  $\mathcal{V}_{L_m}^\bullet$ .** Denote by  $\mathcal{V}_{L_m}^\bullet$  the oscillatory module corresponding to the Lagrangian submanifold  $\xi = mx$ . Local sections of  $\mathcal{V}_{L_m}^\bullet$  are sums

$$(11.1.4) \quad v = \sum_{k \in \mathbb{Z}} v_k$$

where  $v_k$  is a local section of  $\mathcal{V}_{m\frac{x^2}{2}+kx}^\bullet$  on  $\widetilde{M}$ . In other words,  $v_k$  is an  $\Omega_{\mathbb{K}}$ -form on  $\widetilde{M}$  with coefficients in  $\widehat{\mathcal{V}}$  (Definition 9.5). The connection  $\nabla_{\mathcal{V}}$  is given by (cf. 9.4.4)

$$(11.1.5) \quad \nabla_{\mathcal{V}} v_k = (-\frac{\xi - mx - k}{i\hbar}dx + (\frac{\partial}{\partial x} - \frac{\partial}{\partial \widehat{x}} - \frac{1}{i\hbar}m\widehat{x})dx + (\frac{\partial}{\partial \xi} + \frac{1}{i\hbar}\widehat{x})d\xi)v_k$$

The action of  $\widehat{\mathbb{A}}_M$  is as follows (em cf. 9.4.5):  $\widehat{x}$  by multiplication, and  $\widehat{\xi}$  by  $i\hbar\frac{\partial}{\partial \widehat{x}} + m\widehat{x}$ .

*Remark 11.1.* The component  $v_k$  is an element of the form  $\sigma(x, \xi; x, \xi - mx - k)w_k$  where  $w_k$  is a local section of the module  $\mathbb{V}_{L_m}$  (cf. 9.2). Also note that sums (11.1.4) may be infinite but we require that  $v_k \in \exp(\frac{1}{i\hbar}N_k)\widehat{\mathcal{V}}_\Lambda$  where  $N_k \rightarrow \infty$  as  $|k| \rightarrow \infty$ .

Components  $v_k$  satisfy

$$(11.1.6) \quad v_k(x, \xi) = v_{k+1}(x, \xi + 1) = v_{k-m}(x + 1, \xi).$$

The action of  $\widetilde{\mathbf{G}}_M$  on  $\mathcal{V}_{L_m}$  is as follows:

$$(11.1.7) \quad \sigma(x_1, \xi_1; x_2, \xi_2)v_k = \exp(-\frac{1}{i\hbar}(\frac{mx_1^2}{2} + kx_1 - \frac{mx_2^2}{2} - kx_2))g(x_1, \xi_1; x_2, \xi_2)v_k$$

(cf. 9.4.6). It is easy to see directly that all the structures are compatible with each other (of course this also follows from the fact that the above construction is obtained by applying the general procedure of 15).

## 11.2. The computation of $\mathbb{R}\text{HOM}(\mathcal{V}_{L_0}^\bullet, \mathcal{V}_{L_m}^\bullet)$ .

**11.2.1. Matrices with coefficients in  $\mathcal{S}^\bullet$ .** Let  $e_\Lambda$ , resp.  $E$ , be the free module over  $\Lambda$ , resp.  $\mathbb{K}$ , with generators  $e_k$ ,  $k \in \mathbb{Z}$ . Recall the differential graded algebra  $\mathcal{S}$  from (10.4.1). Put also

$$(11.2.1) \quad \mathcal{S}_\Lambda^\bullet = C^\bullet(\text{MPar}(n), \Lambda)$$

Let

$$(11.2.2) \quad \text{Matr}(\mathcal{S}) = \varprojlim_{N \rightarrow \infty} \text{Hom}(E, \mathcal{S}^\bullet \otimes E) / \exp(\frac{1}{i\hbar}N) \text{Hom}(E, \mathcal{S}_\Lambda^\bullet \otimes E)$$

Let  $\mathbf{E}_{k\ell}$  be the matrix unit, i.e. the homomorphism sending  $e_k$  to  $e_\ell$  and  $e_j$  to zero if  $j \neq k$ .

11.2.2.

**Theorem 11.2.** *The sheaf of complexes  $\mathbb{R}\text{HOM}^\bullet(\mathcal{V}_{L_0}^\bullet, \mathcal{V}_{L_m}^\bullet)$  is quasi-isomorphic to the sheaf of sections of the trivial bundle with fiber  $\text{Matr}(\mathcal{S}^\bullet)$ , with the action of  $\pi_1(M)$  as follows. Let  $\gamma_1$  and  $\gamma_2$  be the two generators of  $\pi_1(M)$ , namely  $\gamma_1$  the loop  $\xi = \xi_0, x = x_0 + t$  and  $\gamma_2$  the loop  $x = x_0, \xi = \xi_0 + t$ . Then for a matrix unit  $\mathbf{E}_{k\ell}$*

$$\gamma_1^q \gamma_2^p : \mathbf{E}_{k\ell} \mapsto \exp\left(\frac{1}{i\hbar}\left(\frac{mq^2}{2} + q(\ell - k)\right)\right) \mathbf{E}_{k+p, \ell+p-mq}$$

*Proof.* First construct the  $\mathcal{A}_M^\bullet$ -free resolution  $\mathcal{R}_{L_0}^\bullet$  of  $\mathcal{V}_{L_0}^\bullet$  as in (10.2.8). Local sections of  $\mathcal{R}_{L_0}^\bullet$  are sums (11.1.4) with the same relations (11.1.6)  $m = 0$ ;  $v_k$  are elements of  $\mathcal{R}_k^\bullet$  on  $\widetilde{M}$  which is constructed exactly as  $\mathcal{R}^\bullet$  in (10.2.8) with the only modification: equation (11.2.3) becomes

$$(11.2.3) \quad \nabla_{\mathbb{P}} v_{0,k} = \frac{1}{i\hbar}(-(\xi + k)dx + \widehat{x}d\xi)v_{0,k}$$

Now, local sections of  $\text{Hom}_{\mathcal{A}_M^\bullet}(\mathcal{R}_{L_0}^\bullet, \mathcal{V}_{L_m}^\bullet)$  are sums  $\sum_{k,\ell} b_{k\ell}$  where

$$b_{k\ell} \in \mathcal{C}_{k\ell}^\bullet;$$

here  $\mathcal{C}_{k\ell}^\bullet$  is the complex (10.3.7) computed for the function

$$(11.2.4) \quad f_{k\ell}(x, \xi) = mx^2 + (\ell - k)x$$

Local sections  $b_{k\ell}$  satisfy the following:

$$(11.2.5) \quad b_{k\ell}(x, \xi) = b_{k, \ell-m}(x+1, \xi) = b_{k+1, \ell+1}(x, \xi+1)$$

(Note that all  $\mathcal{C}_{k\ell}^\bullet$  are identical as graded spaces, with the differential  $d_{k\ell}$  on  $\mathcal{C}_{k\ell}^\bullet$  given by

$$d_{k\ell} = \text{Ad}(\exp(-\frac{1}{i\hbar}(\frac{mx^2}{2} + (\ell - k)x + \frac{m\widehat{x}^2}{2})))d_{00}).$$

The action of the fundamental groupoid is as follows. A path  $\gamma : (x_1, \xi_1) \rightarrow (x_2, \xi_2)$  in  $\widetilde{M}$  preserves each  $\mathcal{C}_{k\ell}^\bullet$  and acts on it by

$$(11.2.6) \quad (\gamma b)_{k\ell}(x_1, \xi_1) = \exp\left(\frac{1}{i\hbar}\left(\frac{mx_1^2}{2} + (\ell - k)x_1 - \frac{mx_2^2}{2} + (\ell - k)x_2\right)\right) b_{k\ell}(x_2, \xi_2)$$

because of (9.4.1) and because

$$\begin{aligned} & (f_{k\ell}(x_1 + \widehat{x}) - f'_{k\ell}(x_2)\widehat{x}) - f_{k\ell}(x_2 + \widehat{x}) - f'_{k\ell}(x_2)\widehat{x} = \\ & = \frac{mx_1^2}{2} + (\ell - k)x_1 - \frac{mx_2^2}{2} - (\ell - k)x_2. \end{aligned}$$

When  $x_2 - x_1 = q$  and  $\xi_2 - \xi_1 = p$ , the right hand side of (11.2.6) becomes

$$(\gamma b)_{k+p, \ell+p-mq}(x, \xi) = \exp\left(\frac{1}{i\hbar}\left(\frac{mq^2}{2} + q(\ell - k)\right)\right) b_{k\ell}(x, \xi).$$

The statement now follows from Theorem 10.7.  $\square$

**Corollary 11.3.** *For  $m > 0$ , the space of horizontal sections of  $\underline{\mathbb{R}\mathrm{HOM}}^\bullet(\mathcal{V}_{L_0}^\bullet, \mathcal{V}_{L_m}^\bullet)$  is  $m$ -dimensional over  $\mathbb{K}$  with the basis*

$$\theta_a = \sum_{q \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \exp\left(\frac{1}{i\hbar}(mq^2 + aq)\right) \mathbf{E}_{k, k+a-qm}$$

where  $a = 0, 1, \dots, m-1$ .

**11.3. The case of sheaves.** Following Tamarkin, we define the category  $D(\mathbf{T}^2)$ . First define the following diffeomorphisms of  $\mathbb{R} \times \mathbb{R}$ :

$$(11.3.1) \quad S_1(x, t) = (x+1, t); \quad S_2(x, t) = (x, t+x);$$

One has

$$(11.3.2) \quad S_2 S_1 = T_1 S_1 S_2; \quad T_1 S_1 = S_1 T_1; \quad T_1 S_2 = S_2 T_1$$

where  $T_1(x, t) = (x, t+1)$ . (In other words, we have an action of the Heisenberg group  $\mathrm{Heis}(3, \mathbb{Z})$  on  $\mathbb{R} \times \mathbb{R}$ .)

Define objects of  $D(\mathbf{T}^2)$  as equivariant objects of  $D(\mathbb{R}^2)$ , *i.e.* objects  $\mathcal{F}$  of  $D(\mathbb{R})^2$  together with isomorphisms

$$(11.3.3) \quad \sigma_1 : \mathcal{F} \xrightarrow{\sim} S_{1*} \mathcal{F}; \quad \sigma_2 : \mathcal{F} \xrightarrow{\sim} S_{2*} \mathcal{F}$$

in  $\mathrm{HOM}_{\mathbb{K}}$  such that

$$(11.3.4) \quad \sigma_2 \sigma_1 \tau_1 = \sigma_1 \sigma_2$$

or more precisely

$$(11.3.5) \quad (T_1 S_1)_* \sigma_2 \cdot T_{1*} \sigma_1 \cdot \tau_1 = S_{2*} \sigma_1 \cdot \sigma_2$$

as morphisms  $\mathcal{F} \rightarrow (S_2 S_1)_* \mathcal{F} = (T_1 S_1 S_2)_* \mathcal{F}$ .

*Example 11.4.* For an integer  $n$ , put

$$(11.3.6) \quad \mathcal{F}_m = \prod_{k \in \mathbb{Z}} \mathcal{F}_{m \frac{x^2}{2} + kx}$$

In fact,

$$(S_1^q S_2^p)(x, t) = (x+q, t+px);$$

$$(S_1^q S_2^p)^* \mathcal{F}_{m \frac{x^2}{2} + kx} = \mathbb{Z}_{\{t+px+m \frac{(x+q)^2}{2} + k(x+q) \geq 0\}} = T_{m \frac{q^2}{2} + kq}^* \mathcal{F}_{m \frac{x^2}{2} + (k+mq+p)x};$$

In other words, if

$$(11.3.7) \quad \mathcal{L}_k = \mathcal{F}_{m \frac{x^2}{2} + kx},$$

then

$$(11.3.8) \quad \mathcal{F}_m = \prod_{k \in \mathbb{Z}} \mathcal{L}_k; \quad (S_1^q S_2^p)^* \mathcal{L}_k = (T_{m \frac{q^2}{2} + kq})^* \mathcal{L}_{k+mq+p}$$

**11.4. Comparison between the categories.** Consider the following automorphisms of the pair  $(\tilde{\mathbf{G}}_{\mathbb{R}^2}, \mathcal{A}_{\mathbb{R}^2})$ . Let  $\sigma(x_1, \xi_1; x_2, \xi_2)$  be as in (11.1.1). Define

$$(11.4.1) \quad (S_1)\sigma(x_1, \xi_1; x_2, \xi_2) = \sigma(x_1 + 1, \xi_1; x_2 + 1, \xi_2).$$

$$(11.4.2) \quad (S_2\sigma)(x_1, \xi_1; x_2, \xi_2) = \exp\left(\frac{1}{i\hbar}(x_1 - x_2)\right)\sigma(x_1, \xi_1 + 1; x_2, \xi_2 + 1);$$

For a section  $a$  of  $\mathcal{A}_{\mathbb{R}^2}$ , define

$$(11.4.3) \quad (S_1a)(x, \xi, \hat{x}, \hat{\xi}) = a(x + 1, \xi, \hat{x}, \hat{\xi}); \quad (S_2a)(x, \xi, \hat{x}, \hat{\xi}) = a(x, \xi + 1, \hat{x}, \hat{\xi})$$

It is easy to see that these maps preserve all the structures, *i.e.* the product on  $\mathcal{A}$ , the composition on  $\tilde{\mathbf{G}}$ , the action of  $\tilde{\mathbf{G}}$  on  $\mathcal{A}$ , and the flat connection up to inner derivations. Therefore for an oscillatory module  $\mathcal{V}^\bullet$  on  $\mathbb{R}^2$ , one can define new oscillatory modules  $S_1^*\mathcal{V}^\bullet$  and  $S_2^*\mathcal{V}^\bullet$  as follows. As differential graded  $\Omega_{\mathbb{K}}^\bullet$ -modules, they are the inverse images of  $\mathcal{V}^\bullet$  under the shifts  $(x, \xi) \mapsto (x + 1, \xi)$  and  $(x, \xi) \mapsto (x, \xi + 1)$ ; the algebra  $\mathcal{A}_{\mathbb{R}^2}$  and the groupoid  $\tilde{\mathbf{G}}_{\mathbb{R}^2}$  act via automorphisms  $S_1, S_2$ . One has

$$(11.4.4) \quad (S_2^p)^*(S_1^q)^*\mathcal{V}_{m\frac{x^2}{2}+kx}^\bullet = \mathcal{V}_{m\frac{x^2}{2}+(k+mq-p)x}^\bullet$$

Note that the central subgroup  $\{T_c | c \in \mathbb{Z}\}$  of  $\text{Heis}(\mathbb{Z})$  acts on  $\text{HOM}(\mathcal{F}_0, \mathcal{F}_m)$ . Therefore the automorphisms  $\sigma_1$  and  $\sigma_2$  generate an action of  $\mathbb{Z}^2$ .

**11.4.1. Matrix units for tori.** Put

$$(11.4.5) \quad f_{m,k}(x) = m\frac{x^2}{2} + kx$$

Let  $m > 0$ . Define the matrix unit  $\mathbf{E}_{k\ell}$  as follows. Let

$$(11.4.6) \quad \mathbf{E}_{f_{m_1,k}, f_{m+m_1,\ell}} \in \text{HOM}_{\mathbb{K}}(\mathcal{F}_{f_{m_1,k}}, \mathcal{F}_{f_{m+m_1,\ell}})$$

be as in (10.5.3). Let  $i_k$ , resp.  $\text{pr}_k$ , be the embedding of, resp. the projection onto, the  $k$ th component in the decomposition in (11.3.6). Define  $\mathbf{E}_{k\ell}$  as the composition

$$i_\ell \circ \mathbf{E}_{f_{m_1,k}, f_{m+m_1,\ell}} \circ \text{pr}_k : \mathcal{F}_{m_1} \rightarrow \mathcal{F}_{m_1\frac{x^2}{2}+kx} \rightarrow \mathcal{F}_{(m+m_1)\frac{x^2}{2}+\ell x} \rightarrow \mathcal{F}_{m+m_1}$$

One has

$$\begin{aligned} \text{HOM}_{\mathbb{K}}(\mathcal{F}_{f_{m_1,k}}, \mathcal{F}_{f_{m+m_1,\ell}}) &= \mathbb{K}\mathbf{E}_{k\ell} \\ \mathbf{E}_{j\ell} &= \mathbf{E}_{jk}\mathbf{E}_{k\ell} \end{aligned}$$

**Proposition 11.5.** *The action of the group  $\mathbb{Z}^2$  on  $\text{HOM}(\mathcal{F}_0, \mathcal{F}_m)$  is as follows.*

$$\sigma_1^q \sigma_2^p \mathbf{E}_{k\ell} = \exp\left(\frac{1}{i\hbar}\left(m\frac{q^2}{2} + (\ell - k)q\right)\right) \mathbf{E}_{\ell+p, k+p-mq}$$

Now let  $m < 0$ . There is a generator

$$(11.4.7) \quad \mathbf{Z}(f_{m_1,k}, f_{m+m_1,\ell}) \in \mathbb{R}^1 \text{Hom}(\mathcal{F}_{f_{m_1,k}}, (T_{-\sup f_{m,\ell-k}})_* \mathcal{F}_{f_{m+m_1,\ell}})$$

obtained as follows. First, to simplify notation, assume  $m_1 = k = 0$ , as well as  $\sup(f_{m,\ell}) = 0$  (the general case follows immediately). Replace  $\mathcal{F}_0 = \mathbb{Z}_{t \geq 0}$  by the complex

$$(11.4.8) \quad \mathbb{Z}_{t < 0} \rightarrow \mathbb{Z}$$

The complex  $\text{Hom}(\mathbb{Z}_{t < 0} \rightarrow \mathbb{Z}, \mathbb{Z}_{t \geq f_{m,\ell}})$  is isomorphic to  $\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(1,-1)} \mathbb{Z}$  and computes  $\mathbb{R} \text{Hom}(\mathcal{F}_{f_{0,0}}, \mathcal{F}_{f_{m,\ell}})$ .

Put

$$(11.4.9) \quad \mathbf{Z}_{f_{m_1,k}, f_{m+m_1,\ell}} = \exp\left(\frac{1}{i\hbar} \sup(f_{m,\ell-k})\right) \mathbf{Z}(f_{m_1,k}, f_{m+m_1,\ell})$$

Define  $\mathbf{Z}_{k\ell}$  as the composition

$$i_\ell \circ \mathbf{Z}_{f_{m_1,k}, f_{m+m_1,\ell}} \circ \text{pr}_k : \mathcal{F}_{m_1} \rightarrow \mathcal{F}_{m_1 \frac{x^2}{2} + kx} \rightarrow \mathcal{F}_{(m+m_1) \frac{x^2}{2} + \ell x} \rightarrow \mathcal{F}_{m+m_1}$$

We have thus defined

$$(11.4.10) \quad \mathbf{E}_{k\ell}(m_2, m_1) \in \text{HOM}_{\mathbb{K}}^0(\mathcal{F}_{m_1}, \mathcal{F}_{m_2}), \quad m_1 \leq m_2;$$

$$(11.4.11) \quad \mathbf{Z}_{k\ell}(m_2, m_1) \in \text{HOM}_{\mathbb{K}}^1(\mathcal{F}_{m_1}, \mathcal{F}_{m_2}), \quad m_1 > m_2;$$

They satisfy

$$(11.4.12) \quad \mathbf{E}_{jk}(m_3, m_2) \mathbf{E}_{k'\ell}(m_2, m_1) = \delta_{kk'} \mathbf{E}_{k\ell}(m_3, m_1);$$

$$(11.4.13) \quad \mathbf{E}_{jk}(m_3, m_2) \mathbf{Z}_{k'\ell}(m_2, m_1) = \delta_{kk'} \mathbf{Z}_{k\ell}(m_3, m_1)$$

if  $m_1 > m_3$  and zero otherwise;

$$(11.4.14) \quad \mathbf{Z}_{jk}(m_3, m_2) \mathbf{E}_{k'\ell}(m_2, m_1) = \delta_{kk'} \mathbf{Z}_{k\ell}(m_3, m_1)$$

if  $m_1 > m_3$  and zero otherwise;

$$(11.4.15) \quad \mathbf{Z}_{jk}(m_3, m_2) \mathbf{Z}_{k'\ell}(m_2, m_1) = 0$$

**Proposition 11.6.** *The action of the group  $\mathbb{Z}^2$  on  $\text{HOM}(\mathcal{F}_0, \mathcal{F}_m)$  is as follows.*

$$\sigma_1^q \sigma_2^p \mathbf{Z}_{k\ell} = \exp\left(\frac{1}{i\hbar} \left(m \frac{q^2}{2} + (\ell - k)q\right)\right) \mathbf{Z}_{\ell+p, k+p-mq}$$

## 12. APPENDIX. METAPLECTIC AND METALINEAR GROUPS

We recall the classical material that is contained, for example, in [18] and in [44].



**12.1. Metalinear groups and metalinear structures.** Recall [18] that the metalinear group is by definition

$$(12.1.1) \quad \text{ML}(n, \mathbb{R}) = \{(g, z) | g \in \text{GL}(n, \mathbb{R}), z^2 = \det(g)\}$$

This is a twofold cover of  $\text{GL}(n, \mathbb{R})$ . There is a morphism

$$(12.1.2) \quad \det^{\frac{1}{2}} : \text{ML}(n, \mathbb{R}) \rightarrow \mathbb{C}^\times; (g, z) \mapsto z.$$

Denote by  $\text{MO}(n)$  the preimage of  $\text{O}(n)$  in  $\text{ML}(n)$ . Let also

$$(12.1.3) \quad \text{MU}(n) = \{(u, \zeta) | u \in \text{U}(n, \mathbb{C}), \zeta^2 = \det(u)\}$$

**Definition 12.1.** Let  $\text{Mp}(2n, \mathbb{R})$  be the universal twofold cover of  $\text{Sp}(2n, \mathbb{R})$ . We call this group *the metaplectic group*.

There is a commutative diagram

$$\begin{array}{ccc} \text{MO}(n) & \longrightarrow & \text{ML}(n, \mathbb{R}) \\ \downarrow & & \downarrow \\ \text{MU}(n) & \longrightarrow & \text{Mp}(2n, \mathbb{R}) \end{array}$$

where the horizontal embeddings are homotopy equivalences.

A metalinear structure on a real vector bundle  $E$  is a lifting of the transition automorphisms  $g_{jk}^E$  to an  $\text{ML}(n, \mathbb{R})$ -valued cocycle  $\tilde{g}_{jk}^E$ . For a real bundle  $E$  with a metalinear structure, the complex line bundle  $\wedge^{\frac{1}{2}} E$  is by definition given by the transition automorphisms  $\det^{\frac{1}{2}}(\tilde{g}_{jk}^E)$ , cf. (12.1.2).

A metaplectic structure on a symplectic vector bundle  $E$  is a lifting of the transition automorphisms  $g_{jk}^E$  to an  $\text{Mp}(n, \mathbb{R})$ -valued cocycle  $\tilde{g}_{jk}^E$ . A metalinear structure on a manifold (resp. a metaplectic structure on a symplectic manifold) is by definition the corresponding structure on its tangent bundle.

**Lemma 12.2.** *A manifold  $X$  has a metalinear structure if and only if  $T^*X$  has a metaplectic structure. If a symplectic manifold has a metaplectic structure then any Lagrangian submanifold of  $M$  has a metalinear structure.*

*Proof.* The obstruction to existence of a metalinear, resp. metaplectic, structure is as follows. Pick any transition isomorphisms  $g_{jk}$  for the tangent bundle. Lift them to a cochain  $\tilde{g}_{jk}$  with values in  $\text{ML}$ , resp. in  $\text{Mp}$ . Then compute the two-cocycle  $a_{jkl} = \tilde{g}_{jk}\tilde{g}_{kl}\tilde{g}_{jl}^{-1}$  with values in  $\mathbb{Z}/2\mathbb{Z}$ . The cohomology class of this cocycle is the obstruction. If  $M = T^*X$ , this cohomology class is determined by its restriction to  $X$ . But on  $X$  the symplectic transition functions  $g_{jk}$  for  $TM$  can be chosen as the image of  $\text{GL}(n)$ -valued transition functions for  $TX$  under the embedding  $\text{GL} \rightarrow \text{Sp}$ . This proves the first statement of the Lemma. Now, for a Lagrangian submanifold  $L$  of  $M$ , the transition isomorphisms for  $TM|_L$  are cohomologous to an  $\text{Mp}$ -valued cocycle  $p_{jk} : g_{jk} = h_j p_{jk} h_k^{-1}$ . Lift  $h_j$  to  $\text{Mp}(2n)$  somehow. Put

$$(12.1.4) \quad \tilde{p}_{jk} = \tilde{h}_j^{-1} \tilde{g}_{jk} \tilde{h}_k.$$

This is a cocycle cohomologous to  $\tilde{g}_{jk}|L$ . It takes values in the preimage of the subgroup of  $\mathrm{Sp}(2n)$  consisting of matrices preserving the Lagrangian submanifold  $L_0 = \{\xi = 0\}$ . The image of this cocycle under the projection to  $\mathrm{GL}$  via  $\mathrm{ML}$  is a cocycle defining the bundle  $TX$ .  $\square$

## 12.2. The Maslov class of a Lagrangian submanifold.

**12.2.1. The case  $c_1(M) = 0$ .** The cohomology class of the two-cocycle  $a_{jkl}$  constructed as in the proof of Lemma 12.2 above but when we use the universal cover  $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$  instead of  $\mathrm{Mp}(n)$ . This is now a class in  $H^2(M, \mathbb{Z})$  that represents  $c_1(M)$ , the first Chern class of  $TM$  viewed as a complex bundle after we reduce the structure group  $\mathrm{Sp}$  to the maximal compact subgroup  $U(n)$ . Indeed,  $\widetilde{\mathrm{Sp}}$  is homotopy equivalent to

$$\tilde{U}(n) = \{(u, x) | u \in U(n), x \in \mathbb{R}, \det(u) = e^{2\pi i x}\}.$$

The proof of Lemma 12.2 applied to this case establishes the following fact.

Consider the group

$$(12.2.1) \quad \widetilde{\mathrm{GL}}(n, \mathbb{R}) = \{(g, x) | x \in \mathbb{R}; \det(g) = e^{2\pi i x}\}$$

(Of course  $\widetilde{\mathrm{GL}}$ , unlike  $\tilde{U}$  or  $\widetilde{\mathrm{Sp}}$ , has nothing to do with universal covers).

**Lemma 12.3.** *A trivialization of  $c_1(M)$  defines a  $\widetilde{\mathrm{GL}}(n)$ -structure on any Lagrangian submanifold  $L$  of  $M$ , i.e. a lifting of the transition automorphisms of  $TL$  to a  $\widetilde{\mathrm{GL}}(n)$ -valued cocycle.*

Assume that  $L$  is oriented. Then there is another  $\widetilde{\mathrm{GL}}(n)$ -structure on  $L$ , due to the fact that  $\mathrm{SL}(n)$  is a subgroup of  $\widetilde{\mathrm{GL}}(n)$ . The two liftings differ by a class in  $\lambda(L) \in H^1(L, \mathbb{Z})$ . We will call this class *the Maslov class of an oriented Lagrangian submanifold* of a symplectic manifold  $M$  with a trivialization of  $c_1(M)$ .

**12.2.2. The case  $2c_1(M) = 0$ .** Now consider the group

$$(12.2.2) \quad \tilde{U}^{(2)}(n) = \{(g, x) | g \in U(n); x \in \mathbb{R}; \det(g)^2 = e^{2\pi i x}\}$$

Note that

$$(12.2.3) \quad \{(g, x) | x \in \mathbb{R}; \det(g)^2 = e^{2\pi i x}\} \xrightarrow{\sim} \mathrm{GL}(n, \mathbb{R}) \times \mathbb{Z}$$

Arguing exactly as before, we get

**Lemma 12.4.** *A trivialization of  $2c_1(M)$  defines a  $\mathrm{GL}(n) \times \mathbb{Z}$ -structure on any Lagrangian submanifold  $L$  of  $M$ .*

Projecting to  $\mathbb{Z}$ , we get a class  $\mu(L) \in H^1(L, \mathbb{Z})$ . We call  $\mu(L)$  *the Maslov class of a Lagrangian submanifold* of a symplectic manifold  $M$  with a trivialization of  $2c_1(M)$ .

Note that

$$(12.2.4) \quad \mu(L) = 2\lambda(L)$$

for a trivialization of  $c_1$ , the induced trivialization of  $2c_1$ , and an oriented  $L$ .

*Remark 12.5.* Let  $\tilde{\Lambda}(n)$  be the universal cover of the Lagrangian Grassmannian  $\Lambda(n)$ . Define the group  $\widetilde{\mathrm{Sp}}^{(2)}(2n, \mathbb{R})$  by the condition that the following square be Cartesian.

$$\begin{array}{ccc} \widetilde{\mathrm{Sp}}^{(2)}(2n, \mathbb{R}) & \longrightarrow & \tilde{\Lambda}(n) \\ \downarrow & & \downarrow \\ \mathrm{Sp}(2n, \mathbb{R}) & \longrightarrow & \Lambda(n) \end{array}$$

Then  $\tilde{U}^{(2)}$  is a homotopy equivalent subgroup of  $\widetilde{\mathrm{Sp}}^{(2)}(2n, \mathbb{R})$ .

*Example 12.6.* For  $n = 1$ ,  $U(1) \xrightarrow{\sim} S^1$ ; also  $\Lambda(1) \xrightarrow{\sim} S^1$ . Under these identifications, the projection  $U(1) \rightarrow \Lambda(1)$  becomes the map  $\zeta \mapsto \zeta^2$ .

**12.3. The groups  $\mathrm{Sp}^N$ .** Here we use definitions and notation from [44]. For  $N \geq 1$ , let  $\Lambda^N(n)$  be the universal  $N$ -fold cover of  $\Lambda(n)$ . Define the group  $\mathrm{Sp}^N(2n, \mathbb{R})$  by requiring the following diagram to be Cartesian:

$$\begin{array}{ccc} \mathrm{Sp}^N(2n, \mathbb{R}) & \longrightarrow & \Lambda^N(n) \\ \downarrow & & \downarrow \\ \mathrm{Sp}(2n, \mathbb{R}) & \longrightarrow & \Lambda(n) \end{array}$$

In other words,  $\mathrm{Sp}^N(2n) = \widetilde{\mathrm{Sp}}^{(2)}(2n)/(\mathbb{Z}/N)$ . Define also

$$U^N(n) = \{(u, \zeta) | u \in U(n), \zeta \in \mathbb{C}, \det(u)^2 = \zeta^N\} = \tilde{U}^{(2)}/(\mathbb{Z}/N)$$

This is a subgroup of  $\mathrm{Sp}^N(n)$  and the embedding is a homotopy equivalence.

A  $\mathrm{Sp}^N(2n)$ -structure on  $M$  is the same as a trivialization of  $2c_1(M)$  in  $H^2(M, \mathbb{Z}/N)$ .

The universal  $N$ -fold cover of  $\mathrm{Sp}(2n)$  is a subgroup of  $\mathrm{Sp}^{2N}(2n)$ . In particular, the metaplectic group  $\mathrm{Mp}(2n)$  is a subgroup of  $\mathrm{Sp}^4(2n)$ . The latter is generated by  $\mathrm{Mp}(2n)$  and the central subgroup  $\{\pm 1, \pm i\}$ . The intersection of the two is  $\{\pm 1\}$ , the kernel of  $\mathrm{Mp} \rightarrow \mathrm{Sp}$ .

The following makes sense for any  $N$ . We fix  $N = 4$  just to fix the notation for the rest of the paper.

**Definition 12.7.** a) Define  $P(n, \mathbb{R})$  as the subgroup of  $\mathrm{Sp}(2n, \mathbb{R})$  consisting of pairs  $(A, z)$  where  $A = \begin{bmatrix} b & a \\ 0 & (b^{-1})^t \end{bmatrix}$  is a symplectic matrix. In other words,  $P(n)$  is the subgroup of  $\mathrm{Sp}(2n)$  consisting of matrices preserving the Lagrangian submanifold  $L_0 = \{\hat{\xi} = 0\}$ .

b) Define  $\mathrm{MPar}(n, \mathbb{R})$  as the subgroup of  $\mathrm{Sp}^4(2n, \mathbb{R})$  consisting of pairs  $(A, z)$  where  $A = \begin{bmatrix} b & a \\ 0 & (b^{-1})^t \end{bmatrix}$  is a symplectic matrix,  $z$  is a complex

number, and  $\det(b)^2 = z^4$ . In other words, this is the lifting to  $\mathrm{Sp}^4(2n)$  of  $P(n)$ .

**Lemma 12.8.** *a)  $\mathrm{MPar}(n, \mathbb{R}) \xrightarrow{\sim} P(n, \mathbb{R}) \times \{\pm 1, \pm i\}$*

*b) If a symplectic manifold  $M$  has an  $\mathrm{Sp}^4$  structure and  $L$  is a Lagrangian submanifold then formulas (12.1.4) define an  $\mathrm{MPar}(n)$ -valued cocycle cohomologous to the transition isomorphisms of  $TM|L$ .*

*c) If  $M$  has a real polarization then it has an  $\mathrm{Sp}^4(2n)$ -structure.*

**Definition 12.9.** The projection of the cohomology class from Lemma 12.8, b) to  $H^1(L, \mathbb{Z}/4\mathbb{Z})$  is called *the Maslov class* of  $L$ .

When the trivialization of  $2c_1(M)$  modulo 4 comes from a trivialization of  $2c_1(M)$  then the Maslov class defined above is equal to  $\exp(\frac{i\pi}{2}\mu(L))$  that was defined in 12.2.2.

### 13. APPENDIX. THE ALGEBRAIC METAPLECTIC REPRESENTATION

Most of the material of this section is contained in [48]. Recall the algebra  $\mathcal{A}$  from 4.1 and the  $\mathcal{A}$ -module from Definition 9.5. In this section we give an interpretation of this module in terms of the metaplectic representation.

**13.1. Symmetries of the deformation quantization algebra of a formal neighborhood.** Any continuous automorphism  $g$  of  $\hat{\mathbb{A}}$  induces a symplectic linear transformation  $g_0$  of  $\mathbb{C}^{2n}$ . Denote by  $G$  the group of those  $g$  whose linear part  $g_0$  preserves the real structure. We have

$$(13.1.1) \quad G = \mathrm{Sp}(2n, \mathbb{R}) \ltimes \exp(\mathfrak{g}_{\geq 1})$$

Define the central extension

$$(13.1.2) \quad \tilde{\mathbf{G}} = \exp\left(\frac{1}{i\hbar}\mathbb{C} \oplus \mathbb{C}\right) \times \widetilde{\mathrm{Sp}}(2n, \mathbb{R}) \ltimes \exp(\tilde{\mathfrak{g}}_{\geq 1})$$

where  $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$  is the universal cover of  $\mathrm{Sp}(2n, \mathbb{R})$ . One has an exact sequence

$$(13.1.3) \quad 1 \rightarrow \mathbb{Z} \times \exp\left(\frac{1}{i\hbar}\mathbb{C}[[\hbar]]\right) \rightarrow \tilde{\mathbf{G}} \rightarrow G \rightarrow 1$$

Define also  $P$  to be the subgroup of  $G$  consisting of elements  $g$  whose linear part preserves the Lagrangian subspace

$$(13.1.4) \quad L_0 = \{\hat{\xi}_1 = \dots = \hat{\xi}_n = 0\}$$

Let  $\tilde{\mathbf{P}}$  be the preimage of  $P$  in  $\tilde{\mathbf{G}}$ .

**13.2. The algebraic Fourier transform.** Let  $\hat{y} = (\hat{y}_1, \dots, \hat{y}_n)$  be  $n$  formal variables. For a symmetric real  $n \times n$  matrix  $a$ , put

$$(13.2.1) \quad \mathcal{H}_a^{\hat{y}} = \exp\left(\frac{a\hat{y}^2}{2i\hbar}\right) \hat{\mathbb{C}}[[\hat{y}, \hbar]]((e^{\frac{c}{i\hbar}} | c \in \mathbb{C}))$$

Here

$$(13.2.2) \quad \widehat{\mathbb{C}}[[\widehat{y}, \hbar]] = \left\{ \sum_{k=-N}^{\infty} v_k | v_k \in \mathbb{C}[[\widehat{y}]]((\hbar))_k \right\}$$

with respect to the grading (3.1.3); for any vector space  $V$ , we define

$$(13.2.3) \quad V((e^{\frac{c}{i\hbar}} | c \in \mathbb{C})) = \left\{ \sum_{k \in \mathbb{N}; \operatorname{Re}(c_k) \rightarrow +\infty} e^{\frac{c_k}{i\hbar}} v_k \right\},$$

$v_k \in V$ . In particular, the operator of multiplication by  $\hbar$  is automatically invertible.

Put also For a nondegenerate  $a$ , define the Fourier transform (cf. [28])

$$(13.2.4) \quad F : \mathcal{H}_a^{\widehat{y}} \xrightarrow{\sim} \mathcal{H}_{-a^{-1}}^{\widehat{\eta}}$$

as follows. Heuristically,

$$(13.2.5) \quad (Ff)(\widehat{\eta}) = \frac{e^{-\frac{\pi i n}{4}}}{(2\pi i \hbar)^{n/2}} \int e^{\frac{\widehat{y}\widehat{\eta}}{i\hbar}} f(\widehat{y}) d\widehat{y};$$

To give the above formula a rigorous meaning, put

$$\begin{aligned} F(f(\widehat{y}) \exp(\frac{a\widehat{y}^2}{2i\hbar}))(\widehat{\eta}) &= f(i\hbar \frac{\partial}{\partial \widehat{\eta}}) F(\exp(\frac{a\widehat{\eta}^2}{2i\hbar})) = \\ f(i\hbar \frac{\partial}{\partial \widehat{\eta}}) \frac{e^{-\frac{\pi i n}{4}}}{\det(\sqrt{ia})} \exp(\frac{-a^{-1}\widehat{\eta}^2}{2i\hbar}) &= \frac{e^{-\frac{\pi i p(a)}{2}}}{\det \sqrt{|a|}} f(i\hbar \frac{\partial}{\partial \widehat{\eta}}) \exp(\frac{-a^{-1}\widehat{\eta}^2}{2i\hbar}) \end{aligned}$$

Here  $p(a)$  is the number of positive eigenvalues of  $a$ . We used the branch of the square root for which  $\sqrt{ix} > 0$  if  $ix > 0$ ; it is defined on the complex plane with the line  $\{ix < 0, x \in \mathbb{R}\}$  removed. The final term in the above chain of equalities can be viewed as the definition of the first term.

*Remark 13.1.* The definition of the Fourier transform  $F$  extends to elements of the form

$$(13.2.6) \quad \mathbf{f}(\widehat{y}) = \exp(\frac{a\widehat{y}^2}{2i\hbar} + i\widehat{y}\widehat{z}) f(\widehat{y})$$

where  $a$  is nondegenerate and  $\widehat{z}$  is another formal parameter:

$$(13.2.7) \quad F(\mathbf{f})(\widehat{\eta}) = F(\exp(\frac{a\widehat{y}^2}{2i\hbar}) f(\widehat{y}))(\widehat{\eta} + \widehat{z})$$

One has

$$(13.2.8) \quad F^2(\mathbf{f})(\widehat{y}) = i^{-n} \mathbf{f}(-\widehat{y}); \quad F\widehat{y}F^{-1} = i\hbar \frac{\partial}{\partial \widehat{\eta}}; \quad Fi\hbar \frac{\partial}{\partial \widehat{y}} F^{-1} = -\widehat{\eta}$$

**13.3. The two-dimensional case.** For the readers convenience, we first present the case  $n = 1$ .

$$(13.3.1) \quad \mathcal{H} = \bigoplus_{a \in \mathbb{R}} \mathcal{H}_a^{\hat{x}} \bigoplus_{a \in \mathbb{R}} F\mathcal{H}_a^{\hat{x}} / \sim$$

where

$$(13.3.2) \quad Ff(\hat{x}) \exp\left(\frac{a\hat{x}^2}{2i\hbar}\right) \sim \frac{e^{-\frac{\pi i}{2}p(a)}}{\sqrt{|a|}} f(i\hbar \frac{\partial}{\partial \hat{x}}) \exp\left(-\frac{a^{-1}\hat{x}^2}{2i\hbar}\right)$$

for  $a \neq 0$ . Here  $p(a) = 1$  if  $a > 0$  and  $p(a) = 0$  otherwise.

**13.3.1. The action of  $\hat{\mathbb{A}}$  on  $\mathcal{H}$ .** The algebra  $\hat{\mathbb{A}}$  acts on the space  $\mathcal{H}$  as follows. If  $\mathbf{f}$  is in the first summand in (13.3.1), then  $\hat{x}$  acts on it by multiplication and  $\hat{\xi}$  by  $i\hbar \frac{\partial}{\partial \hat{x}}$ , the latter defined by

$$\frac{\partial}{\partial \hat{x}}(\exp\left(\frac{a\hat{x}^2}{2i\hbar}\right)f(\hat{x})) = \exp\left(\frac{a\hat{x}^2}{2i\hbar}\right)\left(\frac{\partial}{\partial \hat{x}} + a\hat{x}\right)f(\hat{x}).$$

As for  $F\mathbf{f}$ ,  $\hat{x}$  sends it to  $-i\hbar F \frac{\partial}{\partial \hat{x}}\mathbf{f}$  and  $\hat{\xi}$  sends it to  $F\hat{x}\mathbf{f}$ .

**13.3.2. Some operators on  $\mathcal{H}$ .** The operator  $F : \mathcal{H} \rightarrow \mathcal{H}$ . Define for  $\mathbf{f}(\hat{x}) = \exp\left(\frac{a\hat{x}^2}{2i\hbar}\right)f(\hat{x})$

$$F : \mathbf{f} \mapsto F\mathbf{f} \mapsto i^{-1}\mathbf{f}(-\hat{x})$$

The operator  $\exp\left(\frac{a\hat{x}^2}{2i\hbar}\right) : \mathcal{H} \rightarrow \mathcal{H}$ . 1)

$$\exp\left(\frac{a\hat{x}^2}{2i\hbar}\right) : \exp\left(\frac{c\hat{x}^2}{2i\hbar}\right)f(\hat{x}) \mapsto \exp\left(\frac{(a+c)\hat{x}^2}{2i\hbar}\right)f(\hat{x})$$

for  $c \in \mathbb{R}$ ;

2)

$$F \exp\left(\frac{c\hat{x}^2}{2i\hbar}\right)f(\hat{x}) \mapsto \frac{e^{-\frac{\pi i}{2}p(c)}}{\sqrt{|c|}} f\left(-i\hbar \frac{\partial}{\partial \hat{x}} + a\hat{x}\right) \exp\left(\frac{(a-c^{-1})\hat{x}^2}{2i\hbar}\right)$$

for  $c \neq 0$ ;

3)

$$F \exp\left(\frac{c\hat{x}^2}{2i\hbar}\right)f(\hat{x}) \mapsto iFf(\hat{x} - ai\hbar \frac{\partial}{\partial \hat{x}}) \frac{e^{-\frac{\pi i}{2}(p(c)+p(\frac{-c}{1-ac}))}}{\sqrt{|1-ac|}} \exp\left(\frac{c}{1-ac} \frac{\hat{x}^2}{2i\hbar}\right)$$

for  $c \neq a^{-1}$ . These maps preserve the equivalence relation and therefor define operators on  $\mathcal{H}$ .

**13.3.3. The action of  $\mathrm{Sp}^4(2, \mathbb{R})$  on  $\mathcal{H}$ .** The group  $\mathrm{Sp}^4(2, \mathbb{R})$  acts by the algebraic version of the metaplectic representation that we are going to describe next.

**13.4. The metaplectic projective representations of  $\mathrm{SL}_2(\mathbb{R})$ .** Define the action of generators of  $\mathrm{SL}_2(\mathbb{R})$  by exactly the same formula as the usual metaplectic representation

$$(13.4.1) \quad T: \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \mapsto \exp\left(\frac{a\hat{x}^2}{2i\hbar}\right); \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mapsto F;$$

$$(13.4.2) \quad \begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix} \mapsto T_b; \quad (T_b f)(x) = \frac{1}{\sqrt{\det(b)}} f(b^{-1}x)$$

The corresponding representation of  $\mathfrak{sl}(2)$  is given by

$$(13.4.3) \quad X_- = \frac{\hat{x}^2}{2i\hbar}; \quad H = \frac{\hat{x}\hat{\xi}}{i\hbar} = -\frac{\hat{x} * \hat{\xi}}{i\hbar} - \frac{1}{2}; \quad X_+ = -\frac{\hat{\xi}^2}{2i\hbar}$$

**13.4.1. The Bruhat relations.** The following are well known to be the defining relations of  $\mathrm{SL}_2$  (together with the requirement that  $a \mapsto \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$  is a morphism from the additive group and  $b \mapsto \begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix}$  is a morphism from the multiplicative group).

$$(13.4.4) \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix}$$

$$(13.4.5) \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} b^{-1} & 0 \\ 0 & b \end{bmatrix}$$

$$(13.4.6) \quad \begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ b^{-2}a & 1 \end{bmatrix}$$

$$(13.4.7) \quad \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & a^{-1} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a^{-1} & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$$

for  $a \neq 0$ .

**Proposition 13.2.** *Formulas (13.4.1) define a representation of  $\widetilde{\mathrm{SL}}_2$  in which an element of  $\pi_1(\mathrm{SL}_2)$  acts by  $e^{\frac{\pi i}{2}c}$  where  $c$  is its image in  $\pi_1(\Lambda) \xrightarrow{\sim} \mathbb{Z}$ .*

*Proof.* All the Bruhat relations except (13.4.7) are true for operators  $T(g)$  defined in (13.4.1), whereas

**Lemma 13.3.**

$$\begin{aligned} & T\left(\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}\right) T\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) T\left(\begin{bmatrix} 1 & a^{-1} \\ 0 & 1 \end{bmatrix}\right) = \\ & = \frac{\sqrt{|a|}}{\sqrt{a}} e^{\frac{\pi i}{2}p(a)} T\left(\begin{bmatrix} a^{-1} & 0 \\ 0 & a \end{bmatrix}\right) T\left(\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}\right) \end{aligned}$$

□

**13.5. The case of a general  $n$ .** Now define

$$(13.5.1) \quad \mathcal{H} = \bigoplus_{I \subset \{1, \dots, n\}} \bigoplus_a F_{I,J} \mathcal{H}^{\hat{x}} / \sim$$

where  $a$  runs through all symmetric  $n \times n$  real matrices and the equivalence relation is defined as follows. For every subset  $K$  of  $\{1, 2, \dots, n\}$ , define

$$(13.5.2) \quad F_K : \bigoplus_a F_I \mathcal{H}^{\hat{x}} \rightarrow \bigoplus_a F_{I \Delta K} \mathcal{H}^{\hat{x}}$$

(where  $\Delta$  stand for the symmetric difference) as follows: if  $L$  is the complement of  $I \cap K$ , then

$$(13.5.3) \quad (F_K F_I \mathbf{f})(\hat{x}_{I \cap K}, \hat{x}_L) = i^{-|I \cap K|} F_{I \Delta K} \mathbf{f}(-\hat{x}_{I \cap K}, \hat{x}_L)$$

Let  $J$  be the complement of  $I$ .

$$(13.5.4) \quad \mathbf{f}(\hat{x}_I, \hat{x}_J) = \exp\left(\frac{a\hat{x}_I^2 + b\hat{x}_I\hat{x}_J + c\hat{x}_J^2}{2i\hbar}\right) f(\hat{x}_I, \hat{x}_J)$$

such that  $a_I$  is a nondegenerate symmetric matrix. Then

$$(13.5.5) \quad F_K F_I \mathbf{f} \sim F_K \frac{\exp(-\frac{\pi i}{2} p(a))}{\sqrt{\det(|a|)}} f(i\hbar \frac{\partial}{\partial \hat{x}_I}) \exp\left(\frac{-a^{-1}(\hat{x}_I + b\hat{x}_J)^2}{2i\hbar}\right)$$

for all  $K$ .

**13.5.1. Operators on  $\mathcal{H}$ .** Clearly, the operators  $F_K$  (13.5.2) preserve the equivalence relation and therefore descend to  $\mathcal{H}$ .

**13.5.2. The action of  $\hat{\mathbb{A}}$  on  $\mathcal{H}$ .** The algebra  $\hat{\mathbb{A}}$  acts on the space  $\mathcal{H}$  as follows. On the summand  $F_I \mathcal{H}_a^{\hat{x}}$ ,

$$(13.5.6) \quad \hat{x}_j F_I \mathbf{f} = -F_I i\hbar \frac{\partial}{\partial \hat{x}_j} \mathbf{f}, j \in I; \quad \hat{x}_j F_I \mathbf{f} = F_I \hat{x}_j \mathbf{f}, j \notin I;$$

$$(13.5.7) \quad \hat{\xi}_j F_I \mathbf{f} = F_I i\hbar \frac{\partial}{\partial \hat{x}_j} \mathbf{f}, j \notin I; \quad \hat{\xi}_j F_I \mathbf{f} = F_I \hat{x}_j \mathbf{f}, j \in I.$$

**13.5.3. The action of  $\mathrm{Sp}^4(2n)$  on  $\mathcal{H}$ .** This action is exactly as described in 13.3.3. In particular,  $\mathrm{Sp}^4(2n, \mathbb{R})$  acts by the metaplectic representation as in 13.4:

$$(13.5.8) \quad T: \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \mapsto \exp\left(\frac{a\hat{x}^2}{2i\hbar}\right); \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mapsto F;$$

more generally, let  $\mathbf{F}_I$  be the matrix that is the direct sum of  $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$  in coordinates  $\hat{x}_I, \hat{\xi}_I$  and the identity matrix in the rest of the Darboux



coordinates maps to  $F_I$ ;

$$(13.5.9) \quad \begin{bmatrix} b & 0 \\ 0 & {}_t b^{-1} \end{bmatrix} \mapsto T_b; (T_b f)(x) = \frac{1}{\sqrt{\det(b)}} f(b^{-1}x)$$

*Remark 13.4.* The construction of  $\mathcal{H}$  mimics very closely the construction of the orbit of 1 under the action of  $\mathrm{Sp}^4(2n) \ltimes \mathbb{C}[\widehat{x}, \widehat{\xi}]$  on the space of distributions via differential operators and the standard metaplectic representation.

**Lemma 13.5.** *Assign to  $F_I \exp(\frac{a\widehat{x}^2}{2i\hbar})f(\widehat{x}) \in \mathcal{H}$  the Lagrangian subspace  $\mathbf{F}_I(\{\widehat{\xi} = a\widehat{x}\})$  where  $\mathbf{F}_I$  was defined after (13.5.8). This is a well-defined map  $\mathcal{H} \rightarrow \Lambda(n)$  where  $\Lambda(n)$  is the Grassmannian of Lagrangian subspaces in  $\mathbb{R}^{2n}$ . The space  $\mathcal{H}$  is identified with the space of finitely supported sections of a  $\widetilde{\mathbf{G}}$ -equivariant vector bundle on  $\Lambda(n)$ .*

The Lagrangian Grassmannian is a homogeneous space of  $\widetilde{\mathbf{G}}$  via the projection  $\widetilde{\mathbf{G}} \rightarrow \mathrm{Sp}^4 \rightarrow \mathrm{Sp}$ . In fact,

$$\Lambda(n) \xrightarrow{\sim} \widetilde{\mathbf{G}}/\widetilde{\mathbf{P}}.$$

**Lemma 13.6.** *The lines  $\mathbb{C}F_I \exp(\frac{a\widehat{x}^2}{2i\hbar})$  where  $a$  runs through real symmetric  $n \times n$  matrices and  $I$  through subsets of  $\{1, \dots, n\}$  form a line subbundle of  $\mathcal{H}$  which is isomorphic to the bundle on  $\Lambda(n)$  determined by the Čech one-cohomology class  $(-1)^{\mu_L}$  where  $\mu_L$  is the generator of  $H^1(\Lambda(n), \mathbb{Z})$  (the Maslov class).*

**Lemma 13.7.** *The actions described in 13.5.2 and 13.5.3 turn  $\mathcal{H}$  into an  $\mathcal{A}$ -module.*

### 13.6. The algebraic metaplectic representation as an induced module.

**Proposition 13.8.**

$$\mathcal{H} \xrightarrow{\sim} \widehat{\mathcal{V}} = \mathcal{A} \widehat{\otimes}_B \widehat{\widehat{\mathbf{V}}}_{\mathbb{K}}$$

(cf. 9.2.1).

## 14. APPENDIX. TWISTED BUNDLES AND GROUPOIDS

**14.1. Charts and cocycles.** Suppose we have a manifold  $X$  with two sheaves of groups  $\underline{C} \subset \underline{G}$  where  $\underline{C}$  is constant and central in  $\underline{G}$ . Consider a class  $c \in H^2(X, \underline{C})$ . A  $\underline{G}$ -bundle on  $X$  twisted by  $c$  is given by an equivalence class of  $g_{ij} \in \underline{G}(U_i \cap U_j)$  for an open cover  $X = \cup U_i$  such that

$$(14.1.1) \quad g_{ij}g_{jk} = c_{ijk}g_{ik}$$

where  $c_{ijk}$  is a Čech cocycle representing  $c$ . Two data  $g_{ij}$  and  $g'_{ij}$  are equivalent if

$$(14.1.2) \quad g_{ij} = h_i g'_{ij} h_j^{-1} b_{ij}$$

for some common refinement of the two covers, where  $h_i \in \underline{G}(U_i)$  and  $b_{ij} \in \underline{C}(U_i \cap U_j)$ . Note that this definition makes all  $\underline{C}$ -bundles equivalent.

By a *chart* we mean a map  $T \rightarrow X$  where  $T$  is any topological space. A *good collection of charts* on  $X$  is a collection of charts  $T \rightarrow X$ ,  $T \in \mathcal{T}$ , such that for every  $T_0, \dots, T_p$  in  $\mathcal{T}$ , every one-cocycle on  $T_0 \times_X \dots \times_X T_p$  with values in the pullback of  $\underline{G}$ , and every one- or two-cocycle with values in the pullback of  $\underline{C}$ , can be trivialized.

**Lemma 14.1.** *For any good collection of charts and any twisted bundle, one can define*

$$(14.1.3) \quad c_{TT'T''} \in \underline{C}(T \times_X T' \times_X T''); \quad g_{TT'} \in \underline{G}(T \times_X T')$$

*satisfying*

$$(14.1.4) \quad c_{TT'T''} c_{TT''T'''} = c_{TT'T'''} c_{T'T''T'''} \quad \text{and}$$

$$(14.1.5) \quad g_{TT'} g_{T'T''} = c_{TT'T''} g_{TT''}$$

*in such a way that, if  $T_i$  are a good open cover, then  $c_{T_i T_j T_k}$  is cohomologous to  $c_{ijk}$  and  $g_{T_i T_j}$  is equivalent to  $g_{ij}$ . The choice is unique up to equivaklence in the following sense:*

$$(14.1.6) \quad c_{TT'T''} = c'_{TT'T''} b_{TT'} b_{T'T''} b_{TT''}^{-1}; \quad g_{TT'} = h_T g'_{TT'} h_{T'}^{-1} b_{TT'}$$

*for some  $b_{TT'} \in \underline{C}(T \times T')$  and  $h_T \in \underline{G}(T)$ .*

*Proof.* Consider inverse images on charts  $T$  of an open cover  $\{U_i\}$  of  $X$ . Let

$$c_{ijk} = \alpha_{ij}(T) \alpha_{jk}(T) \alpha_{ik}(T)^{-1}$$

be a trivialization of  $c$  on  $T$ . Choose trivializations

$$g_{ij} \alpha_{ij}(T)^{-1} = h_i(T) h_j(T)^{-1}$$

on  $T$  and

$$\alpha_{ij}(T) \alpha_{ij}(T') = \beta_i(T, T') \beta_j(T, T')^{-1}$$

where  $\alpha_{ij}, \beta_{ij}$  are sections of  $\underline{C}$  and  $h_i$  are sections of  $\underline{G}$ . Put

$$(14.1.7) \quad c_{TT'T''} = \beta_i(T, T') \beta_i(T', T'') \beta_i(T, T'')^{-1}$$

and

$$(14.1.8) \quad g_{TT'} = h_i(T)^{-1} h_i(T') \beta_i(T, T')$$

The relations above show that these do not depend on  $i$ . □

**14.2. The groupoid of a twisted  $G$ -bundle.** Let  $G$  be a Lie group and  $\underline{G}$  the sheaf of smooth  $G$ -valued functions. Let  $C$  be a central subgroup of  $G$  and  $\underline{C}$  the sheaf of locally constant  $C$ -valued functions. Consider a  $\underline{C}$ -valued two-cohomology class represented by a cocycle  $c_{ijk}$  and twisted  $G$ -bundle represented by a  $\underline{G}$ -valued cochain  $g_{jk}$ .

Define a groupoid on  $X$  as follows.

For  $x_0$  and  $x_1$  in  $X$ , define the set  $\tilde{\mathbf{G}}_{x_0, x_1}$ . Let  $\gamma : [0, 1] \rightarrow X$  be a smooth map. View it as a chart that we denote by  $T$ . An element of  $\tilde{\mathbf{G}}_{x_0, x_1}$  is a class of an element  $g_T \in G$  with respect to the following equivalence

relation. Consider two charts  $T$  and  $T'$  representing two smooth maps  $\gamma, \gamma' : [0, 1] \rightarrow X$  and a homotopy  $\sigma : [0, 1]^2 \rightarrow X$  such that  $\sigma(0, s) = x_0$ ,  $\sigma(s, t) = x_1$ ,  $\sigma(t, 0) = \gamma(t)$ , and  $\sigma(t, 1) = \gamma'(t)$ . We will view  $\sigma$  as a chart  $S$ . We call  $S$  a homotopy between  $S$  and  $S'$ . Now generate the equivalence relation by the following.

$$(14.2.1) \quad g_T \sim (g_{TT'} c_{TT'S}^{-1})(x_0) g_{T'} (g_{TT'} c_{TT'S}^{-1})(x_1)^{-1}$$

**Lemma 14.2.** *Let  $S$  be a homotopy between  $T$  and  $T'$ ,  $S'$  a homotopy between  $T'$  and  $T''$ , and  $S''$  a homotopy between  $T$  and  $T''$ . If we denote the right hand side of (14.2.1) by  $a(S)g_T$ , then  $a(S)a(S') = \langle c, [\Sigma] \rangle a(S'')$  where  $\Sigma$  is the sphere formed by  $S, S'$ , and  $S''$ .*

*Proof.* We have

$$a(S)a(S')g_T =$$

$$g_{TT'} g_{T'T''} c_{TT'S}^{-1} c_{T'T''S'}^{-1}(x_0) g_{T''} (g_{TT'} g_{T'T''} c_{TT'S}^{-1} c_{T'T''S'}^{-1}(x_1))^{-1}$$

The right hand side is equal to

$$(g_{TT''} c_{TT'T''} c_{TT'S}^{-1} c_{T'T''S'}^{-1})(x_0) g_{T''} (g_{TT''} c_{TT'T''} c_{TT'S}^{-1} c_{T'T''S'}^{-1})(x_1)^{-1};$$

therefore

$$a(S)a(S') = \frac{c_{TT'T''} c_{TT''S''}}{c_{TT'S} c_{T'T''S'}}(x_0) \left( \frac{c_{TT'T''} c_{TT''S''}}{c_{TT'S} c_{T'T''S'}}(x_1) \right)^{-1} a(S'')$$

Applying the acyclicity condition to the quadruple of charts  $TT'T''S$ , we get

$$\frac{c_{TT'T''} c_{TT''S''}}{c_{TT'S} c_{T'T''S'}} = \frac{c_{TT''S''} c_{T'T''S'}}{c_{TT''S} c_{T'T''S'}}$$

Applying the same condition to  $TT''SS''$  and then to  $SS'S''T''$ , we replace the right hand side with

$$\frac{c_{TSS''} c_{T''SS'}}{c_{T''S'S''} c_{T'SS'}} = \frac{c_{SS'S''} c_{TSS''}}{c_{T'SS'} c_{T''S'S''}}.$$

But  $T \times_X S \times_X S'' = T$ ,  $T' \times_X S \times_X S' = T'$ , and  $T'' \times_X S' \times_X S'' = T''$ . Therefore the values of  $c_{TSS''}$ , etc. at  $x_0$  and  $x_1$  are the same. Therefore

$$a(S)a(S') = c_{SS'S''}(x_0) c_{SS'S''}(x_1)^{-1} a(S'')$$

But

$$c_{SS'S''}(x_0) c_{SS'S''}(x_1)^{-1} = \langle c, [\Sigma] \rangle$$

for any two-cocycle  $c$ . To see this, note that the left hand side is 1 for any coboundary  $c$ . On the other hand, if we enlarge  $S, S', S''$  a little bit to make them an open cover of  $\Sigma$ , take an element  $a$  of  $C$ , and define  $c_{SS'S''}(x_0) = a$ ,  $c_{SS'S''}(x_1) = 1$ , the result will be  $a = \langle c, [\Sigma] \rangle$ .  $\square$

**Corollary 14.3.** *There is an epimorphism*

$$(14.2.2) \quad \tilde{\mathbf{G}}_{x_0, x_1} \rightarrow \pi_1(x_0, x_1)$$

*When  $x_0 = x_1 = x$ , the kernel of this epimorphism is isomorphic to  $G/\langle c, \pi_2(X) \rangle$ .*

**14.2.1. Example: the holonomy groupoid of a vector bundle.** Let  $E$  be a real oriented vector bundle of rank  $N$ . Let  $G = \mathrm{SO}_N(\mathbb{R})$  and  $\tilde{G} = \mathrm{Spin}_N(\mathbb{R})$  its universal cover. Reduce the structure group of  $E$  to  $G$  using a Riemannian metric. Let  $\widetilde{\mathrm{Isom}}(E)_{x_0, x_1}$  be the set of equivalence classes of data  $(\gamma, u_t)$  where  $\gamma : [0, 1] \rightarrow X$  is a smooth map,  $\gamma(0) = x_0$ ,  $\gamma(1) = x_1$ , and  $u_t : E_{\gamma(t)} \xrightarrow{\sim} E_{\gamma(0)}$  a metric-preserving linear isomorphism smoothly depending on  $t$  and satisfying  $u_0 = \mathrm{Id}$ . An equivalence between  $(\gamma, u_t)$  and  $(\gamma', u'_t)$  is a smooth map  $\sigma : [0, 1] \times [0, 1] \rightarrow X$  such that  $\sigma(0, s) = x_0$ ,  $\sigma(1, s) = x_1$ ,  $\sigma(t, 0) = \gamma(t)$ ,  $\sigma(t, 1) = \gamma'(t)$ , and a linear metric-preserving isomorphism  $v_{t,s} : E_{\sigma(t,s)} \xrightarrow{\sim} E_{x_0}$  smooth in  $(t, s)$ , such that  $v_{0,s} = \mathrm{Id}$ ,  $v_{t,0} = u_t$ ,  $v_{t,1} = u'_t$ , and  $v_{1,s} = u_1 = u'_1$ .

Lift the transition isomorphisms  $g_{ij}^E$  of  $E$  to some  $\tilde{g}_{ij}$ . Put  $c_{ijk} = \tilde{g}_{ij}\tilde{g}_{jk}\tilde{g}_{ik}^{-1}$ . This cocycle represents the second Stiefel-Whitney class  $w_2(E)$ . Note that the groupoid  $\widetilde{\mathrm{Isom}}(E)$  is isomorphic to the groupoid  $\tilde{\mathbf{G}}'$  constructed from the twisted bundle defined by  $\tilde{g}_{ij}, c_{ijk}$ . In fact, note that for the charts  $T$  and  $S$  defined by maps  $\gamma$  and  $\sigma$ , there is a natural lifting  $\tilde{g}_{TS}$  of  $g_{TS}$ . Namely,  $\tilde{g}_{TS}(\gamma(t))$  is the class of the path  $g_{TS}(\gamma(\tau))$ ,  $0 \leq \tau \leq t$ . Similarly with  $\tilde{g}_{ST'}$ . Identify with  $\tilde{G}$  the set of equivalence classes of  $(\gamma, u_t)$  with fixed  $\gamma$  (and with  $\sigma(t, s) = \gamma(t)$  in the definition of the equivalence). Now, given an equivalence  $\sigma, v$  between  $\gamma, u$  and  $\gamma', u'$ ,  $g_T \in \tilde{G}$  gets identified with  $\tilde{g}_{TS}\tilde{g}_{ST'} = \tilde{g}_{TT'}c_{TST'}$ .

**Corollary 14.4.** *There is an epimorphism*

$$(14.2.3) \quad \widetilde{\mathrm{Isom}}(E)_{x_0, x_1} \rightarrow \pi_1(x_0, x_1)$$

*and every preimage is a homogeneous space  $\mathrm{Spin}(N, \mathbb{R})/\langle w_2(E), \pi_2(X) \rangle$ . (We identify  $\mathbb{Z}/2$  with the center of  $\mathrm{Spin}(N, \mathbb{R})$ ).*

**14.2.2. Connections on twisted bundles.** As in 14.2, let  $G$  be a simply-connected (pro) Lie group and  $\underline{G}$  the sheaf of smooth  $G$ -valued functions. Let  $C$  be a central subgroup of  $G$  and  $\underline{C}$  the sheaf of smooth  $C$ -valued functions. In addition, fix some algebra  $\mathcal{A}$  on which  $G$  acts by automorphisms. Consider a twisted bundle defined by the data  $(g_{ij}, c_{ijk})$ . A connection in this twisted bundle is a collection of  $\mathcal{A}$ -valued forms on  $U_i$  such that

$$\mathrm{Ad}_{g_{ij}}(d + A_j) = d + A_i$$

on every  $U_{ij}$ . Here  $\mathrm{Ad}_g(d) = -dg \cdot g^{-1}$ . Note that, because  $c_{ijk}$  are locally constant and central,  $\mathrm{Ad}_{g_{ij}}\mathrm{Ad}_{g_{jk}}(d + A_k) = \mathrm{Ad}_{g_{ik}}(d + A_k)$ , so the conditions above are consistent on  $U_{ijk}$ . The curvature  $R = dA_i + A_i^2$  is a well-defined  $\mathcal{A}$ -valued two-form.

**14.2.3. The flat connection up to inner derivations.** Here we will construct a flat connection up to inner derivations on the associated bundle of algebras  $\mathcal{A}$  compatible with the action of the groupoid  $\tilde{\mathbf{G}}$  of a twisted bundle (cf. 14.2). We will start from a flat connection on the twisted bundle itself.

First define special coordinate charts on  $\tilde{\mathbf{G}}$  as follows. Fix:

- two open charts  $U_0$  and  $U_1$  of  $X$ ;
- two points  $x_0^* \in U_0$  and  $x_1^* \in U_1$ ;
- a path  $\gamma$  from  $x_0^*$  to  $x_1^*$  in  $X$ ;
- smooth maps  $\tau_0 : [0, 1] \times U_0 \rightarrow U_0$  and  $\tau_1 : [0, 1] \times U_1 \rightarrow U_1$ ,  
 $\tau_0(0, x_0) = x_0$ ,  $\tau_0(1, x_0) = x_0^*$ ,  $\tau_1(0, x_1) = x_1$ ,  $\tau_1(1, x_1) = x_1^*$ .

For every  $x_0 \in U_0$  and  $x_1 \in U_1$ , we will denote the path  $t \mapsto \tau_0(t, x_0)$  by  $\tau_{x_0}$  and the path  $t \mapsto \tau_1(t, x_1)$  by  $\tau_{x_1}$ . For the data as above, we construct a chart  $T$  in  $\tilde{\mathbf{G}}$  as a map

$$U_0 \times U_1 \rightarrow \tilde{\mathbf{G}}; (x_0, x_1) \mapsto \tau_{x_0} \circ \gamma \circ \tau_{x_1} : x_0 \rightarrow x_1$$

(the composition of paths).

Now consider a flat connection in our twisted bundle. In a local trivialization, on any open chart  $W$ , we write  $\nabla_{\mathcal{V}} = d + A_W$ . We can identify a local section of  $\tilde{\mathbf{G}}$  on  $T$  with a  $\tilde{G}$ -valued function  $g_T(x_0, x_1)$  on  $U_0 \times U_1$ .

**Definition 14.5.**

$$\alpha(g_T) = -dg_T \cdot g_T^{-1} - A_0 + \text{Ad}_{g_T}(A_1)$$

where  $A_0 = \pi_0^*(A_{U_0})$  and  $A_1 = \pi_1^*(A_{U_1})$ ;

$$R = dA_0 + A_0^2.$$

**Lemma 14.6.** *The above formulas define a flat connection up to inner derivations on the associated bundle of algebras  $\mathcal{A}$  compatible with the action of  $\tilde{\mathbf{G}}$ .*

## 15. APPENDIX. MODULES ASSOCIATED TO LAGRANGIAN SUBMANIFOLDS AND LAGRANGIAN DISTRIBUTIONS

For any Lagrangian submanifold  $L$  of a symplectic manifold  $M$  with a given  $\text{Sp}^4$  structure we constructed a bundle of modules  $\hat{\mathbb{V}}_L$  with a flat connection  $\nabla_{\mathbb{V}}$  (cf. 9.2.2). This is a bundle of  $\hat{\mathbb{A}}_M$ -modules, and the connections  $\nabla_{\mathbb{V}}$  and  $\nabla_{\mathbb{A}}$  are compatible. In particular, denote by  $\mathbb{A}_M$  the sheaf of algebras of horizontal sections of  $\nabla_{\mathbb{A}}$  and by  $\mathbb{V}_L$  the sheaf of horizontal sections of  $\nabla_{\mathbb{V}}$ . Then  $\mathbb{V}_L$  is a sheaf of  $\mathbb{A}_M$ -modules.

Now apply the same construction to  $L$  but instead of  $M$  take a tubular neighborhood of  $L$  and identify it with the tubular neighborhood of  $L$  in  $T^*L$  by Darboux-Weinstein theorem. Use the  $\text{Sp}^4$  structure provided by the Lagrangian polarization by fibers of  $T^*L$  (cf. Lemma 12.8). We get another  $\mathbb{A}_M$ -module that we denote by  $\mathbb{V}_L^{(0)}$ .

**Lemma 15.1.**  $\mathbb{V}_L^{(0)} \xrightarrow{\sim} |\Omega|_L^{\frac{1}{2}}$  where  $|\Omega|_L^{\frac{1}{2}}$  stands for the bundle of half-densities.  $\mathbb{V}_L$  is isomorphic to  $\mathbb{V}_L^{(0)}$  twisted by the  $\{\pm 1, \pm i\}$ -valued Maslov class of  $L$ .

We denote this class by  $\exp(\frac{\pi i}{2}\mu(L))$ . Note that  $\mu(L)$  can be chosen as a  $\mathbb{Z}$ -valued cocycle only if  $2c_1(M) = 0$ .

**15.1. The asymptotic construction of Hörmander and Maslov.** As we have seen in 9.2.2, the oscillatory module  $\mathcal{V}_L^\bullet$  is induced from the module of forms with coefficients in  $\widehat{\mathbb{V}}$ . But it is the twisted version of the latter module that serves as an asymptotic version of the classical construction of Lagrangian distributions with wave front  $L$ .

Put

$$(15.1.1) \quad \mathbb{V}_{L,\mathbb{K}} = \mathbb{K} \widehat{\otimes} \mathbb{V}_L = \left\{ \sum_{k=0}^{\infty} e^{\frac{1}{i\hbar} c_k} v_k \mid v_k \in \mathbb{V}_L; c_k \in \mathbb{R}; c_k \rightarrow \infty \right\}$$

**Definition 15.2.** Assume  $M = T^*X$ . Let  $\mathbb{V}_{L,\mathbb{K}}^\eta$  be the twist of the sheaf  $\mathbb{V}_{L,\mathbb{K}}$  by the Čech cohomology class  $\exp(-\frac{1}{i\hbar}\eta) \in H^1(L, \exp(\frac{1}{i\hbar}\mathbb{R}))$  where  $\eta$  is the class of the form  $\xi dx|_L$ .

Let  $X = \cup U_\alpha$  is a small open cover. Let  $L = \cup W_\gamma$  be a refinement of the cover  $L = \cup (T^*U_\alpha \cap L)$ . In particular, a choice is made of  $\gamma \mapsto \alpha = \alpha(\gamma)$  such that  $W_\gamma \subset T^*U_\alpha \cap L$ .

**15.1.1. Quantization procedure.** First let us review our deformation quantization picture in the case  $M = T^*X$ . First, we have the sheaf of algebras  $\mathbb{A}_{T^*X}$ . It can be described by products  $*_\alpha$  on  $C^\infty(T^*U_\alpha)[[\hbar]]$

$$(15.1.2) \quad a *_\alpha b = \sum_{k=0}^{\infty} (i\hbar)^k P_{\alpha,k}(a, b)$$

and by transition functions

$$(15.1.3) \quad G_{\alpha\beta}(a) = \sum_{k=0}^{\infty} (i\hbar)^k T_{\alpha\beta,k}(a)$$

where  $P_{\alpha,k}$  are bilinear bidifferential expressions,  $T_{\alpha\beta,k}$  are differential operators,  $P_{\alpha,0}(f, g) = fg$ ,  $P_{\alpha,1}(f, g) = \frac{1}{2}\{f, g\}$ , and  $T_{\alpha\beta,0}(f) = f$ . One has  $G_{\alpha\beta}(a *_\beta b) = G_{\alpha\beta}(a) *_\alpha G_{\alpha\beta}(b)$ . Actually  $G_{\alpha\beta}$  can be made the identity automorphisms, but this is not necessarily the most natural choice.

The sheaf of modules  $\mathbb{V}_L^\eta$  is described by the action

$$(15.1.4) \quad a *_\gamma f = \sum_{k=0}^{\infty} (i\hbar)^k Q_{\gamma,k}(a, f)$$

where  $f \in |\Omega|^{\frac{1}{2}}(W_\gamma)$  and  $a \in C^\infty(U_{\alpha(\gamma)})$ , and by the transition functions

$$(15.1.5) \quad H_{\gamma\delta}(f) = \exp(-\frac{1}{i\hbar}\eta_{\gamma\delta}) \sum_{k=0}^{\infty} (i\hbar)^k S_{\gamma\delta,k}(f)$$

where  $Q_{\gamma,k}$  are bidifferential and  $S_{\gamma\delta,k}$  are differential. Moreover,  $Q_{\gamma\delta,0}(a, f) = af$  and

$$(15.1.6) \quad S_{\gamma\delta,0}(f) = \exp(\frac{\pi i}{2} \mu_{\gamma\delta}(L)) f.$$

One has

$$a *_{\gamma} (b *_{\gamma} f) = (a *_{\alpha(\gamma)} b) *_{\gamma} f$$

and

$$S_{\gamma\delta}(a *_{\delta} f) = T_{\alpha(\gamma)\alpha(\delta)}(a) *_{\gamma} S_{\gamma\delta}(f)$$

Again, all higher  $S_{\gamma\delta,k}$  can be made zero, but this is not the most natural choice.

Let  $C_{\text{poly}}^{\infty}$  denote functions on  $T^*X$  that are polynomial on fibers. A *quantization procedure* is the following.

1) For any  $\alpha$ , a map

$$(15.1.7) \quad \text{Op}_h^{\alpha} : C_{\text{poly}}^{\infty}(T^*(U_{\alpha})) \rightarrow \mathcal{D}(U_{\alpha}, |\Omega|_X^{\frac{1}{2}})$$

such that

$$\text{Op}_h^{\alpha}(a)\text{Op}_h^{\alpha}(b) = \text{Op}_h^{\alpha}(a *_{\alpha} b)$$

and

$$\text{Op}_h^{\alpha}(G_{\alpha\beta}(a)) = \text{Op}^{\beta}(a)$$

on  $U_{\alpha} \cap U_{\beta}$ . (We can ask for exact equalities, not for asymptotic equalities like we use below, when  $a$  and  $b$  are polynomial).

2) A map

$$(15.1.8) \quad u_h^{\gamma} : |\Omega|_c^{\frac{1}{2}}(W_{\gamma}) \rightarrow |\Omega|_c^{\frac{1}{2}}(U_{\alpha(\gamma)})$$

for all  $\hbar > 0$ , such that

$$\text{Op}_h^{\alpha(\gamma)}(a)u_h^{\gamma}(f) - \sum_{k=0}^N (i\hbar)^k u_h^{\gamma}(Q_{\gamma,k}(a, f)) = O(h^{N+1})$$

and

$$u^{\gamma}(f) - \sum_{k=0}^N (i\hbar)^k u_h^{\delta}(S_{\gamma,\delta,k}(f)) = O(h^{N+1})$$

for all  $N$ .

Let us recall how a quantization procedure is carried out. For every  $\gamma$  choose a *phase function* for  $L \cap W_{\gamma}$  as follows. Let  $\theta = (\theta_1, \dots, \theta_k)$  be a coordinate system on  $\mathbb{R}^k$ . Choose a coordinate system  $x = (x_1, \dots, x_n)$  on  $U_{\alpha(\gamma)}$ . Choose a *phase function* for  $L \cap W_{\gamma}$ , i.e. a function  $\varphi(x, \theta)$  such that

$$(15.1.9) \quad L \cap W_{\gamma} = \{(\xi, x) | \exists \theta \text{ such that } \xi = \varphi_x(x, \theta) \text{ and } \varphi_{\theta}(x, \theta) = 0\}$$

Here  $\varphi_x$  and  $\varphi_{\theta}$  stand for partial derivatives. We assume that the  $n \times (n+k)$  matrix  $(\varphi_{xx}, \varphi_{x\theta})$  is nondegenerate.

*Example 15.3.* Let  $n = 1$ . Assume that  $L = \{\xi = \varphi'(x)\}$ . Then we can choose  $k = 0$  and  $\varphi = \varphi(x)$ . Now let  $L = \{x = -\psi'(\xi)\}$ . Then we can take  $k = 1$  and  $\varphi(x, \theta) = x\theta + \psi(\theta)$ .

*Example 15.4.* More generally, one can always subdivide the coordinates into two groups and write  $x = (x_1, x_2)$ ;  $\xi = (\xi_1, \xi_2)$  so that  $L \cap W_\gamma$  will be of the form

$$(15.1.10) \quad \xi_1 = F_{x_1}(x_1, \xi_2); \quad x_2 = -F_{\xi_1}(x_1, \xi_2)$$

In this case one can take  $\varphi(x_1, x_2, \theta) = x_2\theta + F(x_1, \theta)$ .

Note that the condition that the matrix of second derivatives is nondegenerate means that  $\theta$  in (15.1.9) is unique and therefore  $L \cap W_\gamma$  can be identified with  $\{(x, \theta) | \varphi_\theta(x, \theta) = 0\}$ . (To do that, one may need to pass to a finer open cover). Moreover, we can choose  $n$  out of  $n + k$  coordinates  $x, \theta$  so that they will be coordinates on  $\{\varphi_\theta = 0\}$ . Namely, we can take any  $n$  coordinates such that the corresponding square submatrix of  $(\varphi_{xx}, \varphi_{x\theta})$  is nondegenerate. Denote these coordinates by  $z$  and the other  $k$  coordinates by  $\zeta$ . Choose a procedure for extending functions  $f(z)$  to functions on  $\{(x, \theta)\}$ . Namely, extend  $f(z)$  to  $f(z)\rho(z')$  where  $\rho$  is a function with small support near zero and  $\rho(z') = 0$ .

Given a phase function and a compactly supported half-form  $f = f(z)|dz|^{\frac{1}{2}}$ , define  $u_h^\gamma(f)$  as follows. Denote by  $f(x, \theta)$  the extension of  $f(z)$  as above. Then define

$$(15.1.11) \quad u_h(f) = \frac{e^{-\frac{\pi i k}{4}}}{(2\pi\hbar)^{\frac{k}{2}}} \int e^{-\frac{\varphi(x, \theta)}{i\hbar}} f(x, \theta) d\theta |dx|^{\frac{1}{2}}$$

For the sake of completeness let us outline the proof of the fact that this is indeed a quantization procedure as described above (it is contained essentially in [18] and [20], as well as in [38]).

First, as proven in [20], any two local phase functions differ by a coordinate change

$$\varphi(x, \theta) \mapsto \varphi(g(x), h(x, \theta))$$

followed by

$$\varphi(x, \theta) \mapsto \varphi(x, \theta) + \theta_1^2$$

Here  $\theta_1$  is an extra variable, possibly multlidimensional, and  $\theta_1^2$  stands for the sum of squares of variables. So we can assume that our local phase functions are as in Example 15.4, possibly with  $\theta_1^2$  added. We have two choices of subdivision  $x = (x_1, x_2)$ . Namely, for  $W_\gamma$  we will have

$$x_1^\gamma = (x_1, x_2); \quad x_2^\gamma = (x_3, x_4);$$

for  $W_\delta$ ,

$$x_1^\delta = (x_1, x_3); \quad x_2^\delta = (x_2, x_4).$$

Let  $F_\gamma(x_1, x_2, \xi_3, \xi_4)$  and  $F_\delta(x_1, x_3, \xi_2, \xi_4)$  be functions as in Example 15.4. Let us look for functions  $f_\gamma$  and  $f_\delta$  such that (15.1.11) will give the same answer for the charts  $W_\gamma$  and  $W_\delta$ .

$$(15.1.12) \quad \frac{e^{-\frac{\pi i}{4}(k_3+k_4)}}{(2\pi\hbar)^{\frac{k_3+k_4}{2}}} \int e^{-\frac{1}{i\hbar}(x_3\xi_3+x_4\xi_4+F_\gamma(x_1, x_2, \xi_3, \xi_4))} f_\gamma(x_1, x_2, \xi_3, \xi_4) d\xi_3 d\xi_4 =$$



$$= \frac{e^{-\frac{\pi i}{4}(k_2+k_4)}}{(2\pi\hbar)^{\frac{k_2+k_4}{2}}} \int e^{-\frac{1}{i\hbar}(x_2\xi_2+x_4\xi_4+F_\delta(x_1,x_3,\xi_2,\xi_4))} f_\delta(x_1,x_3,\xi_2,\xi_4) d\xi_2 d\xi_4$$

Applying the inverse Fourier transform we get

$$(15.1.13) \quad e^{-F_\gamma} f_\gamma = \frac{e^{-\frac{\pi i}{4}(k_2-k_3)}}{(2\pi\hbar)^{\frac{k_2+k_3}{2}}} \int e^{\frac{1}{i\hbar}(-x_2\xi_2+x_3\xi_3-F_\delta)} f_\delta d\xi_2 dx_3$$

Compute the right hand side by the stationary phase method. The critical points satisfy

$$(15.1.14) \quad x_2 = -\frac{\partial F_\delta}{\partial \xi_2}; \quad \xi_3 = \frac{\partial F_\delta}{\partial x_3}$$

In other words, the critical point  $(\xi_2, x_3)$  is such that  $(x_1, x_2, \xi_1, \xi_2)$  is in  $L$ .

$$(15.1.15) \quad f_\gamma = \epsilon_{\gamma\delta} \exp\left(\frac{1}{i\hbar}((x_3\xi_3 - F_\delta) - (x_2\xi_2 - F_\gamma))\right) \text{mod } \hbar$$

or

$$(15.1.16) \quad f_\gamma = \epsilon_{\gamma\delta} \exp\left(\frac{1}{i\hbar}(\varphi_\delta - \varphi_\gamma)\right) \text{mod } \hbar$$

Here

$$(15.1.17) \quad \epsilon_{\gamma\delta} = e^{-\frac{\pi i}{4}(k_2-k_3)} e^{-\frac{\pi i}{4}(n_-(\gamma,\delta)-n_+(\gamma,\delta))}$$

where  $n_-(\gamma, \delta)$ , resp.  $n_+(\gamma, \delta)$ , is the number of negative, resp. positive, eigenvalues of the matrix of second derivatives of  $F_\delta$  with respect to variables  $\xi_2$  and  $x_3$ . We can re-write (15.1.17) as

$$(15.1.18) \quad \epsilon_{\gamma\delta} = \exp \frac{\pi i}{2}(n_+ - k_2)$$

where, as above,  $n_+$  is the number of positive eigenvalues of the matrix of second derivatives of  $F_\delta$  in variables  $x_2, \xi_3$ .

*Example 15.5.* Let  $F_\gamma(x) = \varphi(x)$  and  $F_\delta(x, \theta) = x\theta - \psi(\theta)$  as in Example 15.3. Let us compute  $\epsilon_{\gamma\delta}$ . One has  $k_2 = 1$ . If  $\varphi_{xx} > 0$  then  $n_2 = 0$ . If  $\varphi_{xx} < 0$  then  $n_2 = 1$ . Therefore

$$\epsilon_{\gamma\delta} = -1 \text{ for } \varphi_{xx} > 0; \quad \epsilon_{\gamma\delta} = 0 \text{ for } \varphi_{xx} < 0.$$

Now compute  $\epsilon_{\delta\gamma}$ . One has  $k_2 = 0$ . If  $\varphi_{xx} > 0$  then  $n_2 = 1$ . If  $\varphi_{xx} < 0$  then  $n_2 = 0$ . Therefore

$$\epsilon_{\delta\gamma} = 1 \text{ for } \varphi_{xx} > 0; \quad \epsilon_{\delta\gamma} = 0 \text{ for } \varphi_{xx} < 0.$$

Now note that  $d\varphi_\gamma = \xi dx|L$  on  $L \cap W_\gamma$  and  $d\varphi_\delta = \xi dx|L$  on  $L \cap W_\delta$ . Therefore, if  $\eta_{\gamma\delta} = \varphi_\gamma - \varphi_\delta$  on  $L \cap W_\gamma \cap W_\delta$ , then  $(\eta_{\gamma\delta})$  represents the cohomology class  $\eta$  corresponding to the De Rham class of  $\xi dx|L$ .

On the other hand, a choice of a local presentation (15.1.10) of  $L$  determines a choice of lifting of transition isomorphisms as in (12.1.4). Indeed, in a tangent space  $T_{(x,\xi)}L$  to a point of  $L \cap W_\gamma$ , let  $\hat{x}, \hat{\xi}$  be formal Darboux

coordinates coming from some local coordinate system. Choose a presentation

$$(15.1.19) \quad \widehat{\xi}_1 = A\widehat{x}_1 + B\widehat{\xi}_2; \quad \widehat{x}_2 = -C\widehat{x}_1 - D\widehat{\xi}_2$$

Construct a symplectic matrix sending  $L_0 = \{\widehat{\xi}_1 = \widehat{\xi}_2 = 0\}$  to  $T_{(x,\xi)}L$  as follows. Let

$$(15.1.20) \quad p(A, B, C, D) : (\widehat{x}_1, \widehat{x}_2, \widehat{\xi}_1, \widehat{\xi}_2) \mapsto (\widehat{x}_1, \widehat{x}_2, A\widehat{x}_1 + B\widehat{\xi}_2, C\widehat{x}_1 + D\widehat{\xi}_2)$$

and

$$(15.1.21) \quad F_{\widehat{x}_2} : (\widehat{x}_1, \widehat{x}_2, \widehat{\xi}_1, \widehat{\xi}_2) \mapsto (\widehat{x}_1, -\widehat{\xi}_2, \widehat{\xi}_1, \widehat{x}_2)$$

One has

$$(15.1.22) \quad T_{(x,\xi)}L = F_{\widehat{x}_2}p(A, B, C, D)L_0$$

Note also that both factors of the right hand side extend automatically to elements in  $\mathrm{Sp}^4$ . Indeed, one can replace  $p(A, B, C, D)$  by the homotopy class of the path  $p(tA, tB, tC, tD)$ ,  $0 \leq t \leq 1$ , and  $F_{\widehat{x}_2}$  by the homotopy class of the path

$$(\widehat{x}_1, \widehat{x}_2, \widehat{\xi}_1, \widehat{\xi}_2) \mapsto (\widehat{x}_1, \widehat{x}_2 \cos t - \widehat{\xi}_2 \sin t, \widehat{\xi}_1, \widehat{x}_2 \sin t + \widehat{\xi}_2 \cos t), \quad 0 \leq t \leq \frac{\pi}{2}$$

It is easy to see that the Maslov class  $\mu$  corresponding to the lifted transition functions thus defined is inverse to the one defined by (15.1.18).

## 16. APPENDIX. TWISTED $A_\infty$ MODULES AND $A_\infty$ FUNCTORS

**16.1. Differential graded categories of  $A_\infty$  functors.** Let  $A$  and  $B$  be two differential graded (DG) categories. For two maps

$$\mathbf{f}, \mathbf{g} : \mathrm{Ob}(A) \rightarrow \mathrm{Ob}(B)$$

define

$$\overline{C}_{\mathbf{f}, \mathbf{g}}^\bullet(A, B) = \prod_{n \geq 1; x_0, \dots, x_n} \mathrm{Hom}^\bullet(A(x_0, x_1) \otimes \dots \otimes A(x_{n-1}, x_n)[n], B(f(x_0), g(x_n)))$$

where the product is taken over all  $x_0, \dots, x_n \in \mathrm{Ob}(A)$ . Put

$$(16.1.1) \quad C_{\mathbf{f}, \mathbf{g}}^\bullet(A, B) = \prod_{x_0 \in \mathrm{Ob}(A)} B(f(x_0), g(x_0)) \times \overline{C}_{\mathbf{f}, \mathbf{g}}^\bullet(A, B)$$

Define the differential  $d$  by

$$(16.1.2) \quad (d_1\varphi)(a_1, \dots, a_{n+1}) = \sum_{j=1}^n (-1)^{\sum_{p \leq j} (|a_p|+1)} \varphi(a_1, \dots, a_j a_{j+1}, \dots, a_{n+1})$$

( $d_1 = 0$  on the first factor of (16.1.1));

$$(16.1.3) \quad (d_2\varphi)(a_1, \dots, a_n) = \sum_{j=1}^n (-1)^{\sum_{p < j} (|a_p|+1)} \varphi(a_1, \dots, d_A a_j, \dots, a_{n+1}) + d_B \varphi(a_1, \dots, a_n)$$

Define

$$d = d_1 + d_2$$

Also define the product

$$\overline{C}_{\mathbf{f},\mathbf{g}}^\bullet(A, B) \otimes \overline{C}_{\mathbf{g},\mathbf{h}}^\bullet(A, B) \rightarrow \overline{C}_{\mathbf{f},\mathbf{h}}^\bullet(A, B)$$

by

(16.1.4)

$$(\varphi \smile \psi)(a_1, \dots, a_{m+n}) = (-1)^{|\psi| \sum_{j=1}^m (|a_j|+1)} \varphi(a_1, \dots, a_m) \psi(a_{m+1}, \dots, a_{m+n})$$

(Note that here  $m$  or  $n$  can be zero, which corresponds to the case of one or both factors lying in the first factor of (16.1.1)).

**Definition 16.1.** An  $A_\infty$  functor  $f : A \rightarrow B$  is a map  $f : \text{Ob}(A) \rightarrow \text{Ob}(B)$  together with an element  $f$  of degree 1 in  $\overline{C}_{\mathbf{f},\mathbf{f}}^\bullet(A, B)$  such that

$$df + f \smile f = 0$$

A *curved*  $A_\infty$  functor is defined the same way but now the cochain  $f$  is allowed to be in  $C_{\mathbf{f},\mathbf{f}}^\bullet(A, B)$ .

**Definition 16.2.** Define the DG category  $\mathbf{C}(A, B)$  as follows. Let objects be  $A_\infty$  functors  $\mathbf{f} : A \rightarrow B$ ; set

$$\mathbf{C}^\bullet(A, B)(f, g) = C_{\mathbf{f},\mathbf{g}}^\bullet(A, B)$$

with the differential

$$\delta\varphi = d\varphi + f \smile \varphi - (-1)^{|\varphi|} \varphi \smile f$$

We define the composition to be the cup product.

Also, define the DG category  $\mathbf{C}_+(A, B)$  the same way as above but with objects being curved  $A_\infty$  functors.

**16.1.1. Equivalence of objects in a DG category.** Let  $\mathbf{C}_1$  be the category with two objects 0 and 1 and two mutually inverse morphisms  $g : 0 \rightarrow 1$  and  $g^{-1} : 1 \rightarrow 0$ .

**Definition 16.3.** Two objects  $\mathbf{x}, \mathbf{y}$  of a DG category  $\mathbf{C}$  are equivalent if there is an  $A_\infty$  functor  $\mathbf{C}_1 \rightarrow \mathbf{C}$  sending 0 to  $\mathbf{x}$  and 1 to  $\mathbf{y}$ .

**Lemma 16.4.** *The relation defined above is an equivalence relation.*

*Proof.* Let  $\mathbf{C}_2$  be the category with three objects 0, 1, 2 and with unique morphism between any two objects. There are functors  $i_{pq} : \mathbf{C}_1 \rightarrow \mathbf{C}_2$  that send 0 to  $p$  and 1 to  $q$ ,  $0 \leq p < q \leq 2$ . If we have one equivalence between  $\mathbf{x}$  and  $\mathbf{y}$  and another between  $\mathbf{y}$  and  $\mathbf{z}$ , then we have a functor

$$(16.1.5) \quad \text{Cobar Bar } k[i_{01}\mathbf{C}_1] *_{k[1]} \text{Cobar Bar } k[i_{12}\mathbf{C}_1] \rightarrow \mathbf{C}$$

that sends 0 to  $\mathbf{x}$ , 1 to  $\mathbf{y}$ , and 2 to  $\mathbf{z}$ . Here  $*$  stands for free product of categories; for any category  $\mathbf{C}$ ,  $k[\mathbf{C}]$  is its linearization, and  $k[1]$  is the category with one object 1 whose ring of endomorphisms is  $k$ . But the left hand side

of (16.1.5) is quasi-isomorphic to  $k[i_{01}\mathbf{C}_1] *_{k[1]} k[i_{12}\mathbf{C}_1] \xrightarrow{\sim} \mathbf{C}_2$ . By the standard transfer of structure [30], [31], [34], we get an  $A_\infty$  morphism  $\mathbf{C}_2 \rightarrow C$  that sends 0 to  $\mathbf{x}$ , 1 to  $\mathbf{y}$ , and 2 to  $\mathbf{z}$ . Composing it with  $i_{02}$ , we get an equivalence between  $\mathbf{x}$  and  $\mathbf{z}$ .  $\square$

**Definition 16.5.** Two  $A_\infty$  functors  $A \rightarrow B$  are equivalent if they are equivalent as objects in  $\mathbf{C}(A, B)$ .

**16.1.2. The bar construction.** The bar construction of a DG category  $A$  is a DG cocategory  $\text{Bar}(A)$  with the same objects where

$$\text{Bar}(A)(x, y) = \bigoplus_{n \geq 0} \bigoplus_{x_1, \dots, x_n} A(x, x_1)[1] \otimes A(x_1, x_2)[1] \otimes \dots \otimes A(x_n, x)[1]$$

with the differential

$$\begin{aligned} d &= d_1 + d_2; \\ d_1(a_1 | \dots | a_{n+1}) &= \sum_{i=1}^{n+1} \pm(a_1 | \dots | da_i | \dots | a_{n+1}); \\ d_2(a_1 | \dots | a_{n+1}) &= \sum_{i=1}^n \pm(a_1 | \dots | a_i a_{i+1} | \dots | a_{n+1}) \end{aligned}$$

The second sum is taken over  $n$ -tuples  $x_1, \dots, x_n$  of objects of  $A$ . The signs are  $(-1)^{\sum_{j < i} (|a_j|+1)}$  for the first sum and  $(-1)^{\sum_{j \leq i} (|a_j|+1)}$  for the second. The comultiplication is given by

$$\Delta(a_1 | \dots | a_n) = \sum_{i=1}^{n-1} (a_1 | \dots | a_i) \otimes (a_{i+1} | \dots | a_n)$$

Dually, for a DG cocategory  $B$  one defines the DG category  $\text{Cobar}(B)$ . The DG category  $\text{Cobar Bar}(A)$  is a cofibrant resolution of  $A$ .

It is convenient for us to work with DG (co)categories without (co)units. For example, this is the case for  $\text{Bar}(A)$  and  $\text{Cobar}(B)$  (we sum, by definition, over all tensor products with at least one factor). Let  $A^+$  be the (co)category  $A$  with the (co)units added, i.e.  $A^+(x, y) = A(x, y)$  for  $x \neq y$  and  $A^+(x, x) = A(x, x) \oplus k\text{Id}_x$ . If  $A$  is a DG category then  $A^+$  is an augmented DG category with units, i.e. there is a DG functor  $\epsilon : A^+ \rightarrow k_{\text{Ob}(A)}$ . (For a set  $I$ ,  $k_I$  is the DG category with the set of objects  $I$  and with  $k_I(x, y) = 0$  for  $x \neq y$ ,  $k_I(x, x) = k$ ). Dually, one defines the DG cocategory  $k^{\text{Ob}(B)}$  and the DG functor  $\eta : k^{\text{Ob}(B)} \rightarrow B^+$  for a DG cocategory  $B$ .

For DG (co)categories with (co)units, define  $A \otimes B$  as follows:  $\text{Ob}(A \otimes B) = \text{Ob}(A) \times \text{Ob}(B)$ ;  $(A \otimes B)((x_1, y_1), (x_2, y_2)) = A(x_1, y_1) \otimes B(x_2, y_2)$ ; the product is defined as  $(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|a_2||b_1|} a_1 a_2 \otimes b_1 b_2$ , and the coproduct in the dual way. This tensor product, when applied to two (co)augmented DG (co)categories with (co)units, is again a (co)augmented DG (co)category with (co)units: the (co)augmentation is given by  $\epsilon \otimes \epsilon$ , resp.  $\eta \otimes \eta$ .

**Definition 16.6.** For DG categories  $A$  and  $B$  without units, put

$$A \otimes B = \text{Ker}(\epsilon \otimes \epsilon : A^+ \otimes B^+ \rightarrow k_{\text{Ob}(A)} \otimes k_{\text{Ob}(B)}).$$

Dually, for DG cocategories  $A$  and  $B$  without counits, put

$$A \otimes B = \text{Coker}(\eta \otimes \eta : k^{\text{Ob}(A)} \otimes k^{\text{Ob}(B)} \rightarrow A^+ \otimes B^+).$$

The following is standard (and straightforward).

**Lemma 16.7.** *There are natural bijections*

$$\text{Ob } \mathbf{C}(A, B) \xrightarrow{\sim} \text{Hom}(\text{Cobar } \text{Bar}(A), B);$$

$$\text{Ob } \mathbf{C}_+(A, B) \xrightarrow{\sim} \text{Hom}(\text{Cobar } \text{Bar}^+(A), B)$$

In other words, an  $A_\infty$  functor  $A \rightarrow B$  is the same as a DG functor  $\text{Cobar } \text{Bar}(A) \rightarrow B$ . A curved  $A_\infty$  functor  $A \rightarrow B$  is the same as a DG functor  $\text{Cobar } \text{Bar}^+(A) \rightarrow B$ .

### 16.1.3. The adjunction formula.

**Lemma 16.8.** *There are natural bijections*

$$\text{Ob } \mathbf{C}(A, \mathbf{C}(B, C)) \xrightarrow{\sim} \text{Hom}_{\text{DGcat}}(\text{Cobar}(\text{Bar}^+(A) \otimes \text{Cobar}(B)), C)$$

$$\text{Ob } \mathbf{C}_+(A, \mathbf{C}_+(B, C)) \xrightarrow{\sim} \text{Hom}_{\text{DGcat}}(\text{Cobar}(\text{Bar}^+(A) \otimes \text{Cobar}^+(B)), C)$$

This (as well as Lemma 16.7) follows from Lemmas 16.10, 16.11, 16.12 below.

**16.1.4. Convolution categories.** Let  $\mathbb{B}$  be a DG cocategory and  $C$  a DG category. For

$$\mathbf{f}, \mathbf{g} : \text{Ob}(\mathbb{B}) \rightarrow \text{Ob}(C),$$

put

$$\overline{\text{Conv}}_{\mathbf{f}, \mathbf{g}}(\mathbb{B}, C) = \prod_{x, y \in \text{Ob}(\mathbb{B})} \text{Hom}^\bullet(\mathbb{B}(x, y), C(\mathbf{f}x, \mathbf{g}y))$$

$$\text{Conv}_{\mathbf{f}, \mathbf{g}}(\mathbb{B}, C) = \prod_{x, y \in \text{Ob}(\mathbb{B})} \text{Hom}^\bullet(\mathbb{B}^+(x, y), C(\mathbf{f}x, \mathbf{g}y))$$

The differential  $d$  is the usual one (induced by the differentials on  $\mathbb{B}$  and  $C$ ). Define the product

$$\text{Conv}_{\mathbf{f}, \mathbf{g}}(\mathbb{B}, C) \otimes \text{Conv}_{\mathbf{g}, \mathbf{h}}(\mathbb{B}, C) \rightarrow \text{Conv}_{\mathbf{f}, \mathbf{h}}(\mathbb{B}, C)$$

$$\overline{\text{Conv}}_{\mathbf{f}, \mathbf{g}}(\mathbb{B}, C) \otimes \overline{\text{Conv}}_{\mathbf{g}, \mathbf{h}}(\mathbb{B}, C) \rightarrow \overline{\text{Conv}}_{\mathbf{f}, \mathbf{h}}(\mathbb{B}, C)$$

as follows. If

$$\Delta b = \sum b^{(1)} \otimes b^{(2)}$$

then

$$(16.1.6) \quad (\varphi \smile \psi)(b) = \sum (-1)^{|\psi||b^{(1)}|} \varphi(b^{(1)}) \psi(b^{(2)})$$

**Definition 16.9.** Define DG categories  $\mathbf{Conv}(\mathbb{B}, C)$  and  $\mathbf{Conv}_+(\mathbb{B}, C)$  as follows. Their objects are maps  $\mathbf{f} : \text{Ob}(\mathbb{B}) \rightarrow \text{Ob}(C)$  together with elements  $f$  of degree one in  $\overline{\text{Conv}}_{\mathbf{f}, \mathbf{f}}(\mathbb{B}, C)$  (resp. in  $\text{Conv}_{\mathbf{f}, \mathbf{f}}(\mathbb{B}, C)$ ) satisfying

$$df + f \smile f = 0.$$

The complex of morphisms between  $f$  and  $g$  is  $\text{Conv}_{\mathbf{f}, \mathbf{g}}(\mathbb{B}, C)$  with the differential

$$\delta\varphi = d\varphi + f \smile \varphi - (-1)^{|\varphi|} \varphi \smile f$$

The composition is the cup product (16.1.6).

**Lemma 16.10.** *There are natural isomorphisms of DG categories*

$$\mathbb{C}(A, B) \xrightarrow{\sim} \mathbf{Conv}(\text{Bar}(A), B)$$

$$\mathbb{C}_+(A, B) \xrightarrow{\sim} \mathbf{Conv}_+(\text{Bar}(A), B)$$

**Lemma 16.11.** *There is a natural bijection*

$$\text{Hom}_{\text{DGcat}}(\text{Cobar}(\mathbb{B}), C) \xrightarrow{\sim} \text{Ob}(\mathbf{Conv}(\mathbb{B}, C))$$

**Lemma 16.12.** *There is a natural isomorphism of DG categories*

$$\mathbf{Conv}(\mathbb{B}_1, \mathbf{Conv}(\mathbb{B}_2, C)) \xrightarrow{\sim} \mathbf{Conv}(\mathbb{B}_1 \otimes \mathbb{B}_2, C)$$

This is a reformulation of a result in [29].

**16.1.5. An  $A_\infty$  functor to  $A_\infty$  modules.** Let  $k$  be a field. By  $\text{dgmod}(k)$  we denote the differential graded category of complex of modules over  $k$ . Let  $R$  be an associative algebra over  $k$ .

**Definition 16.13.** We denote the DG category  $\mathbf{C}(\text{Bar}(R), \text{dgmod}(k))$  by  $\text{Mod}_\infty(R)$  and call it the DG category of  $A_\infty$  modules over  $R$ .

Let  $\mathfrak{X}(R)$  be the category whose objects are pairs  $(\mathcal{B} \xrightarrow{\pi} R, \mathcal{M})$  where  $\mathcal{B}$  is a differential graded algebra,  $\pi$  a quasi-isomorphism of DGA, and  $\mathcal{M}$  a DG module over  $\mathcal{B}$ . A morphism  $(\mathcal{B} \xrightarrow{\pi} R, \mathcal{M}) \rightarrow (\mathcal{B}' \xrightarrow{\pi'} R, \mathcal{M}')$  is a morphism  $\mathcal{B} \rightarrow \mathcal{B}'$  of DGA over  $R$  together with a compatible morphism  $\mathcal{M} \rightarrow \mathcal{M}'$ .

We will construct an  $A_\infty$  functor

$$(16.1.7) \quad \mathfrak{X}(R) \rightarrow \text{Mod}_\infty(R)$$

*Remark 16.14.* An  $A_\infty$  functor from a category  $\mathfrak{X}$  to a DG category  $\mathcal{A}$  is by definition an  $A_\infty$  functor from the linearization of  $\mathfrak{X}$  (viewed as a DG category with zero differential) to  $\mathcal{A}$ .

Define the DG category  $\mathfrak{B}$  as follows. Its objects are the same as objects of  $\mathfrak{X}(R)$  but repeated countably many times, *i.e.* an object of  $\mathfrak{B}$  is a pair  $(\mathbf{x}, n)$  where  $\mathbf{x}$  is an object of  $\mathfrak{X}(R)$  and  $n \in \mathbb{Z}$ . The spaces of morphisms are as follows.

$$(16.1.8) \quad \mathfrak{B}((\mathbf{x}, m), (\mathbf{y}, n)) = 0$$

if  $m < n$  or  $m = n$  but  $\mathbf{x} \neq \mathbf{y}$ . If  $m > n$  and

$$(16.1.9) \quad \mathbf{x} = (\mathcal{B} \xrightarrow{\pi} R, \mathcal{M}), \mathbf{y} = (\mathcal{B}' \xrightarrow{\pi'} R, \mathcal{M}'),$$

then

$$(16.1.10) \quad \mathfrak{B}((\mathbf{x}, m), (\mathbf{y}, n)) = \mathcal{B}' \times \mathfrak{X}(R)(\mathbf{x}, \mathbf{y})$$

By  $a'\mathbf{b}$  we denote the pair  $(a', \mathbf{b})$  where  $a' \in \mathcal{B}'$  and  $\mathbf{b} : \mathbf{x} \rightarrow \mathbf{y}$  is a morphism in  $\mathfrak{X}(R)$ . We denote the underlying morphism  $\mathcal{B} \rightarrow \mathcal{B}'$  also by  $\mathbf{b}$ . Put

$$(16.1.11) \quad \mathfrak{B}((\mathbf{x}, n), (\mathbf{x}, n)) = \mathcal{B}$$

We also denote the right hand side by  $\mathcal{B}\text{Id}_{\mathbf{x}}$ . The composition is given by

$$(16.1.12) \quad (a''\mathbf{b}')(a'\mathbf{b}) = (a''\mathbf{b}'(a'))\mathbf{b}'\mathbf{b}$$

Consider the following right DG module  $\mathbf{M}$  over  $\mathfrak{B}$ . Define

$$\mathbf{M}(\mathbf{x}, n) = \mathcal{M}$$

where  $\mathbf{x} = (\mathcal{B} \rightarrow R, \mathcal{M})$ . Define the action

$$\mathbf{M}(\mathbf{x}, m) \otimes \mathfrak{B}((\mathbf{x}, m), (\mathbf{y}, n)) \rightarrow \mathbf{M}(\mathbf{y}, n)$$

by

$$v \otimes (a'\mathbf{b}) = a'\mathbf{b}(v)$$

Here we denote by  $\mathbf{b}$  the underlying action of the morphism  $\mathbf{b} : \mathbf{x} \rightarrow \mathbf{y}$  on the module, as well as on the algebra.

Define another DG category  $\mathcal{R}$  exactly like  $\mathfrak{B}$  above with the only difference that we put

$$(16.1.13) \quad \mathcal{R}((\mathbf{x}, m), (\mathbf{y}, n)) = R \times \mathfrak{X}(R)(\mathbf{x}, \mathbf{y})$$

instead of (16.1.10) and

$$(16.1.14) \quad \mathfrak{B}((\mathbf{x}, n), (\mathbf{x}, n)) = R$$

instead of (16.1.11). We also denote the right hand side by  $\text{Id}_{\mathbf{x}}R$ . Instead of (16.1.12), the composition is given by the composition is given by

$$(16.1.15) \quad (a''\mathbf{b}')(a'\mathbf{b}) = (a''a')\mathbf{b}'\mathbf{b}$$

The morphisms  $\pi : \mathcal{B} \rightarrow R$  induce a quasi-isomorphism of DG categories  $\mathfrak{B} \xrightarrow{\pi} \mathcal{R}$ . The transfer of structure argument makes  $\mathbf{M}$  a right  $A_{\infty}$  module over  $\mathcal{R}$  as follows. Fix a linear map  $\mathcal{R} \xrightarrow{i} \mathcal{B}$  that is inverse to  $\pi$  at the level of cohomology. (This is where we use the assumption that  $k$  is a field). Fix also homotopies for  $\text{Id}_{\mathfrak{B}} - i\pi$  and for  $\text{Id}_{\mathcal{R}} - \pi i$ . (By this we mean collections of maps  $\mathcal{R}(\mathbf{x}, \mathbf{y}) \rightarrow \mathfrak{B}(\mathbf{x}, \mathbf{y})$ , etc., for any objects  $\mathbf{x}$  and  $\mathbf{y}$ ). From this data one constructs an  $A_{\infty}$  functor  $\mathcal{R} \rightarrow \mathfrak{B}$  which is inverse to  $\pi$  up to equivalence (cf. [30], [31], [34]). Furthermore, the map  $i$  and the homotopies can be chosen to be invariant under the action of  $\mathbb{Z}$  on  $\mathfrak{B}$  and on  $\mathcal{R}$ . Therefore the  $A_{\infty}$  functor is also  $\mathbb{Z}$ -invariant. We denote it by  $\mathbb{T}$ , and the corresponding twisting cochain  $\rho$  by  $\rho_{\mathbb{T}}$ .

This, in turn, defines the desired  $A_\infty$  functor (16.1.7). In fact, for any object  $\mathbf{x} = (\mathcal{B} \rightarrow R, \mathcal{M})$ , the value of this  $A_\infty$  functor on  $\mathbf{x}$  is the underlying complex  $\mathcal{M}$ . For  $g_1, \dots, g_p \in R$ , put

$$(16.1.16) \quad \rho(g_1, \dots, g_p) = \rho_{\mathbb{T}}(g_1 \mathrm{Id}_{\mathbf{x}}, \dots, g_p \mathrm{Id}_{\mathbf{x}})$$

where we view  $\rho_j \mathrm{Id}_{\mathbf{x}}$  as morphisms  $(\mathbf{x}, 0) \rightarrow (\mathbf{x}, 0)$  in  $\mathfrak{B}$ . This makes each  $\mathcal{M}$  an  $A_\infty$  module over  $R$ . Now consider morphisms

$$(16.1.17) \quad \mathbf{x}_0 \xleftarrow{\mathbf{b}_1} \mathbf{x}_1 \xleftarrow{\mathbf{b}_2} \dots \xleftarrow{\mathbf{b}_n} \mathbf{x}_n$$

in  $\mathfrak{X}(R)$ , as well as corresponding morphisms

$$(16.1.18) \quad (\mathbf{x}_0, 0) \xleftarrow{\mathbf{b}_1} (\mathbf{x}_1, 1) \xleftarrow{\mathbf{b}_2} \dots \xleftarrow{\mathbf{b}_n} (\mathbf{x}_n, n)$$

in  $\mathcal{R}$ . Now put

$$\rho(g_1, \dots, g_p) = \sum \pm \rho_{\mathbb{T}}(g_1 \mathrm{Id}_{\mathbf{x}_0}, \dots, g_{p_1} \mathrm{Id}_{\mathbf{x}_0}, \mathbf{b}_1,$$

$$g_{p_1+1} \mathrm{Id}_{\mathbf{x}_1}, \dots, g_{p_2} \mathrm{Id}_{\mathbf{x}_1}, \dots, \mathbf{b}_n, g_{p_n+1} \mathrm{Id}_{\mathbf{x}_n}, \dots, g_p \mathrm{Id}_{\mathbf{x}_n})$$

where the sum is taken over all  $0 \leq p_1 \leq \dots \leq p_n \leq n$ . The sign rule: both  $g_j \mathrm{Id}_{\mathbf{x}_k}$  and  $\mathbf{b}_i$  are treated as odd (the former has degree  $(-1)^{|g_j|+1}$  if  $R$  is graded).

It is straightforward to check that thus defined  $\rho$ , when viewed as a cochain

$$\rho(\mathbf{b}_1, \dots, \mathbf{b}_n) \in \mathrm{Mod}_\infty(R)(\mathcal{M}_0, \mathcal{M}_n),$$

is an  $A_\infty$  functor  $\mathfrak{X}(R) \rightarrow \mathrm{Mod}_\infty(R)$ . (Here  $\mathcal{M}_j$  is the underlying DG module of  $\mathbf{x}_j$ , viewed as a complex).

**16.2. Twisted  $A_\infty$  modules on a space.** Let  $\mathcal{R}$  be a sheaf of algebras on a topological space  $X$ . Fix an open cover  $\mathfrak{U}$  of  $X$ . For two collections  $\mathbf{M} = \{\mathcal{M}_U | U \in \mathfrak{U}\}$  and  $\mathbf{N} = \{\mathcal{N}_U | U \in \mathfrak{U}\}$  of sheaves of  $\mathcal{R}_U$ -modules, define the complex  $C_{\mathbf{M}, \mathbf{N}}^\bullet(\mathfrak{U})$  as follows. Put

$$(16.2.1) \quad C_{\mathbf{M}, \mathbf{N}}^\bullet(\mathfrak{U}) = \prod_{p, q=0}^\infty \prod_{U_0, \dots, U_p \in \mathfrak{U}} \underline{\mathrm{Hom}}^{\bullet-p-q}(\mathcal{R}^{\otimes q}, \underline{\mathrm{Hom}}^\bullet(\mathcal{N}_{U_p}, \mathcal{M}_{U_0}))(U_0 \cap \dots \cap U_p)$$

Define the differentials

$$(16.2.2) \quad (\check{\partial}\varphi)_{U_0 \dots U_{p+1}} = \sum_{j=1}^p (-1)^j \varphi_{U_0 \dots \widehat{U_j} \dots U_{p+1}};$$

$$(16.2.3) \quad (\partial\varphi)(g_1, \dots, g_{q+1}) = (-1)^{p|\varphi|} \sum_{j=1}^q \varphi(g_1, \dots, g_j g_{j+1}, \dots, g_{q+1})$$

for local sections  $g_1, \dots$  of  $\mathcal{R}$ ;

$$(16.2.4) \quad d\varphi = \check{\partial}\varphi + \partial\varphi + d_{\mathcal{M}}\varphi - (-1)^{|\varphi|}\varphi d_{\mathcal{N}}$$

Define also the product

$$(16.2.5) \quad C_{\mathbf{M}, \mathbf{N}}^\bullet(\mathfrak{U}) \otimes C_{\mathbf{N}, \mathbf{P}}^\bullet(\mathfrak{U}) \rightarrow C_{\mathbf{M}, \mathbf{P}}^\bullet(\mathfrak{U})$$



by

$$(\varphi \smile \psi)_{U_0 \dots U_{p_1+p_2}}(g_1, \dots, g_{q_1+q_2}) = (-1)^{|\varphi|p_2+(|\psi|+p_2)q_1} \varphi_{U_0 \dots U_{p_1}}(g_1, \dots, g_{q_1}) \psi_{U_{p_1}, \dots, U_{p_1+p_2}}(g_{q_1+1}, \dots, g_{q_1+q_2})$$

Set

$$(16.2.6) \quad C_{\mathbf{M}, \mathbf{N}}^\bullet(X) = \varinjlim_{\mathfrak{U}} C_{\mathbf{M}, \mathbf{N}}^\bullet(\mathfrak{U})$$

The differential and the cup product are well defined on the above complexes.

**Definition 16.15.** A twisted  $A_\infty$  module  $\mathcal{M}$  over  $\mathcal{R}$  is a collection  $\mathbf{M} = \{\mathcal{M}_U | U \in \mathfrak{U}\}$ , of sheaves of  $\mathcal{R}_U$ -modules together with a cochain  $\rho$  of degree one in  $C_{\mathbf{M}, \mathbf{M}}^\bullet(X)$  such that

$$d\rho + \rho \smile \rho = 0.$$

The DG category  $\text{Tw Mod}_\infty(\mathcal{R})$  has twisted  $A_\infty$  modules as objects. The complex of morphisms between  $\mathcal{M} = (\mathbf{M}, \rho)$  and  $\mathcal{N} = (\mathbf{N}, \sigma)$  is the complex  $C_{\mathbf{M}, \mathbf{N}}^\bullet(X)$  with the differential  $\delta\varphi = d\varphi + \rho \smile \varphi - (-1)^{|\varphi|} \varphi \smile \sigma$ .

The above definition is an extension of the definition of twisted cochains from [40]. Cf. also [39], [3], and [50].

*Remark 16.16.* The DG category of twisted  $A_\infty$  modules is obtained almost *verbatim* as a partial case of the left hand side of Lemma 16.8 . Formally, one could choose  $B$  to be the category with one object whose complex of morphisms is  $\mathcal{R}$ , and  $A = \text{Op}_X$  to be the category of open subsets of  $X$ . More precisely, we perform all the computations as if  $A$  were the category whose objects are open subsets  $U_\alpha$ , and there is one morphism  $U_\alpha \rightarrow U_\beta$  for any two intersecting open subsets. This is not literally true (there may be nonempty intersections  $U_\alpha \cap U_\beta$  and  $U_\beta \cap U_\gamma$  but not  $U_\alpha \cap U_\gamma$ ), but all the formulas work. The above motivation may be given rigorous meaning using the language of \*\*\* as in [14] or in [3].

**16.3. Twisted  $A_\infty$  modules over groupoids.** For  $q \geq 0$ , we use notation  $\mathbb{U} = (U^{(0)}, \dots, U^{(q)})$ . We denote by  $\mathfrak{U}_q$  the set of all such  $\mathbb{U}$  where  $U_j$  is in a given open cover  $\mathfrak{U}$ . For  $p+1$  such  $q$ -tuples  $\mathbb{U}_{j_0}, \dots, \mathbb{U}_{j_p}$ , denote

$$(16.3.1) \quad U_{j_0 \dots j_p}^{(k)} = U_{j_0}^{(k)} \cap \dots \cap U_{j_p}^{(k)}$$

for all  $0 \leq k \leq q$ . Denote also

$$(16.3.2) \quad \mathbb{U}_{j_0 \dots j_p} = (U_{j_0 \dots j_p}^{(0)}, \dots, U_{j_0 \dots j_p}^{(q)}).$$

Here  $\mathcal{M}_{U_0^{(0)}}$  stands for its inverse image under the map

$$\prod_k \cap_j U_j^{(k)} \rightarrow \prod_k U_0^{(k)} \rightarrow U_0^{(0)}$$

Let  $\Gamma$  be an étale groupoid on a manifold  $X$  (in our applications,  $\Gamma = \pi_1(X)$ ). For  $\mathbf{M} = \{\mathcal{M}_U | U \in \mathfrak{U}\}$  and  $\mathbf{N} = \{\mathcal{N}_U | U \in \mathfrak{U}\}$  as in the beginning of 16.2, put

$$C_{\mathbf{M}, \mathbf{N}}^\bullet(\mathfrak{U}, \Gamma) = \prod_{p, q \geq 0} \prod_{\mathbb{U}_0, \dots, \mathbb{U}_p \in \mathfrak{U}_q} \underline{\text{Hom}}^{\bullet-p-q}(\Gamma^{(q)}, \underline{\text{Hom}}^\bullet(\mathcal{N}_{U_p^{(q)}}, \mathcal{M}_{U_0^{(0)}})) \left( \prod_{k=0}^q U_{01\dots p}^{(k)} \right)$$

The differential and the cup product are defined exactly as in (16.2.5), (16.2.3), (16.2.2) (with  $U_j$  replaced by  $\mathbb{U}_j$ ). Define

$$(16.3.3) \quad C_{\mathbf{M}, \mathbf{N}}^\bullet(X, \Gamma) = \varinjlim_{\mathfrak{U}} C_{\mathbf{M}, \mathbf{N}}^\bullet(\mathfrak{U}, \Gamma)$$

**Definition 16.17.** a) Define the DG category  $\text{Tw Mod}_\infty(\Gamma)$  exactly as in Definition 16.15 using complexes  $C_{\mathbf{M}, \mathbf{N}}^\bullet(X, \Gamma)$ .

b) The DG category

$$\text{Tw Mod}_\infty(\Gamma, \Omega_{\mathbb{K}, X}^\bullet)$$

is defined the same way but with  $\mathcal{M}_U$  being  $\Omega_{\mathbb{K}, U}^\bullet$ -modules as in 8.1.

*Remark 16.18.* By

$$\text{Loc}_{\infty, \mathbb{K}}(X)$$

we denote the DG category of  $A_\infty$  representations of the fundamental groupoid  $\pi_1(X)$ . This is the partial case of the above Definition 16.17, a) when  $\Gamma = \pi_1(X)$ , the topology on  $X$  is *discrete*, and the ground ring is  $\mathbb{K}$ . Objects of this DG category are infinity local systems as in 8.4.

**16.3.1. From  $\mathcal{A}_M^\bullet$ -modules with an action of  $\pi_1(M)$  up to inner automorphisms to twisted  $(\Omega_{\mathbb{K}, M}^\bullet, \pi_1(M))$ -modules.** Given two  $\mathcal{A}_M^\bullet$ -modules  $\mathcal{V}^\bullet$  and  $\mathcal{W}^\bullet$  with an action of  $\pi_1(M)$  up to inner automorphisms, consider the standard complex

$$\mathcal{M} = \mathcal{C}^\bullet(\mathcal{V}^\bullet, \mathcal{A}^\bullet, \mathcal{W}^\bullet).$$

As it is shown in 6.2.5,  $\mathcal{M}$  has the following structure.

For a number of open subsets  $U^{(j)}$  indexed by  $j \in J$ , write  $\mathbb{U}_{ij} = (U^{(i)}, U^{(j)})$ . We have constructed:

a) For every  $U^{(0)}$  and  $U^{(1)}$ , an  $\Omega_{\mathbb{K}, U^{(0)} \times U^{(1)}}^\bullet$ -module  $\mathcal{B}_{\mathbb{U}_{01}}$  together with a quasi-isomorphism

$$(16.3.4) \quad \mathcal{B}_{\mathbb{U}_{01}} \rightarrow \mathbb{K} \underline{\pi_1}(M) | (U^{(0)} \times U^{(1)});$$

b) a morphism

$$(16.3.5) \quad p_{01}^* \mathcal{B}_{\mathbb{U}_{01}} \otimes p_{12}^* \mathcal{B}_{\mathbb{U}_{12}} \rightarrow p_{02}^* \mathcal{B}_{\mathbb{U}_{02}}$$

which commutes with the composition on  $\underline{\pi_1}(M)$  under (16.3.4);

c) for any  $U_0^{(j)}$  and  $U_1^{(j)}$ , an isomorphism

$$(16.3.6) \quad \mathbf{b}_{01} : \mathcal{B}_{\mathbb{U}_0} \xleftarrow{\sim} \mathcal{B}_{\mathbb{U}_1}$$

that commutes with (16.3.4) and (16.3.5) and satisfies

$$\mathbf{b}_{01}\mathbf{b}_{12} = \mathbf{b}_{02}$$

on the intersections.

Now repeat the procedure from 16.1.5, together with Remark 16.16, in the above context. First note that the constructions of 16.1.5 can be carried out in the case when  $R$  is a category (and all  $\mathcal{B}$  are DG categories with the same objects). Now act as if  $R$  were the category with objects  $U^{(j)}$ , with

$$R(i, j) = \pi_1(M) | (U^{(i)} \times U^{(j)})$$

and the composition being the one on  $\pi_1$ . Now, let  $\mathrm{Op}_M$  be the category whose objects are open subsets  $U_j$ , exactly as discussed in Remark 16.16. View the data a), b), c) above as a DG functor  $\mathrm{Op}_M \rightarrow \mathfrak{X}(R)$ . Applying formulas from 16.1.5, we get an  $A_\infty$  functor  $\mathrm{Op}_M \rightarrow \mathbf{C}(R, \mathrm{dgmod}(\mathbb{K}))$ , which is the same as an  $\Omega_{\mathbb{K}, M}^\bullet$ -module with a twisted action of  $\pi_1(M)$ .

### 16.3.2. From twisted $(\Omega_{\mathbb{K}, X}^\bullet, \pi_1(X))$ modules to infinity local systems.

Here we extend the construction from 8.4.1. Consider all open covers of the type  $\mathfrak{U} = \{U_x | x \in X\}$ . For an object  $\mathcal{M}$  of  $\mathrm{TwMod}_\infty(\pi_1(X), \Omega_{\mathbb{K}, X}^\bullet)$  choose a cover  $\mathfrak{U}$  as above and define

$$(16.3.7) \quad \mathcal{M}_x = \varinjlim_{U \subset U_x} C^\bullet(U, \mathcal{M}_{U_x})$$

The  $A_\infty$  operators  $T(g_1, \dots, g_n)$  are by definition  $\rho_{\mathbb{U}}(g_1, \dots, g_n)$  where  $g_j \in \pi_1(X)_{x_{j-1}, x_j}$  and  $\mathbb{U} = (U_{x_0}, \dots, U_{x_n})$ . Let us show that different choices of  $\mathfrak{U}$  lead to equivalent infinity local systems (in the sense of Definition 16.5). Choose two covers  $\mathfrak{U}'$  and  $\mathfrak{U}''$ . Apply (16.3.7) to all covers of the form  $\mathfrak{U} = \{U_x | x \in X\}$  where for any  $x$  either  $U_x = U'_x$  or  $U_x = U''_x$ . This data defines an  $A_\infty$  functor  $\mathbb{K}\mathbf{C}_1 \otimes \mathbb{K}\pi_1(X) \rightarrow \mathrm{dgmod}(\mathbb{K})$  (cf. 16.1.1). Let  $\mathbb{K}(0)$ , resp.  $\mathbb{K}(1)$ , be the full subcategory of  $\mathbf{C}_1$  with one object 0, resp. 1. When restricted to  $\mathbb{K}(0)$ , resp. to  $\mathbb{K}(1)$ , our  $A_\infty$  functor coincides with the infinity local system obtained from  $\mathfrak{U}'$ , resp. from  $\mathfrak{U}''$ . By the adjunction formula (Lemma 16.8), the two infinity local systems are equivalent.

*Remark 16.19.* It is easy to modify the above construction and obtain an  $A_\infty$  functor

$$\mathrm{TwMod}(\Omega_{\mathbb{K}, X}^\bullet, \pi_1(X)) \rightarrow \mathrm{Loc}_{\infty, \mathbb{K}}(X).$$

Moreover, the right hand side is a monoidal category up to homotopy, and the assignment  $\mathcal{M}, \mathcal{N} \mapsto \mathbb{R}\mathrm{HOM}(\mathcal{M}, \mathcal{N})$  turns oscillatory modules, as well as  $\Omega_{\mathbb{K}, M}^\bullet$ -modules with an action of  $\pi_1(M)$  up to inner automorphisms, into a category enriched over it. The main reason for this is Lemma 6.17. We will provide the details in a subsequent work.

## 17. APPENDIX. JETS AND TWISTED MODULES

Here we will describe the deformation quantization and the twisted bundle  $\mathcal{H}_M$  in terms of bundles of jets.

**17.1. Jet bundles.** Let  $M$  be any manifold and let  $\mathcal{E}$  be a complex vector bundle of rank  $N$  on  $M$ . Here we recall the construction of the bundle whose fiber at a point  $x$  is the space of jets of sections of  $\mathcal{E}$  at  $x$ . This bundle has the canonical connection; its horizontal sections are determined by sections  $s$  of  $\mathcal{E}$ . The value of such a section at any  $x$  is the jet of  $s$  at  $x$ .

Let  $\{U_\alpha\}$  is an open cover and  $x_\alpha = (x_{\alpha,1}, \dots, x_{\alpha,n})$  a local coordinate system on  $U_\alpha$ . For  $x \in U_\alpha \cap U_\beta$ , we denote by  $x_\alpha$ , resp.  $x_\beta$ , its coordinates in the corresponding coordinate system and write

$$(17.1.1) \quad x_\alpha = g_{\alpha\beta}(x_\beta)$$

Let  $h_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \mathrm{GL}_N$  be the transition isomorphisms of  $\mathcal{E}$ . We identify a local section of  $\mathcal{E}$  on  $U_\alpha \cap U_\beta$  with a  $\mathbb{C}^N$ -valued function in the coordinates  $x_\beta$ .

Let  $\mathbb{C}^N[[\hat{x}]] = \mathbb{C}^N[[\hat{x}_1, \dots, \hat{x}_n]]$ . For  $x \in U_\alpha$  define  $G_{\beta\alpha}(x) : \mathbb{C}^N[[\hat{x}]] \rightarrow \mathbb{C}^N[[\hat{x}]]$  by  $G_{\beta\alpha}(x) : f_\alpha \mapsto f_\beta$  where

$$(17.1.2) \quad f_\beta(\hat{x}) = h_{\alpha\beta}(x_\beta + \hat{x})f_\alpha(g_{\alpha\beta}(x_\beta + \hat{x}) - x_\alpha)$$

It is easy to see that different choices of covers and of local trivializations lead to isomorphic bundles. We denote the bundle defined by (17.1.2) by  $\mathrm{Jets}(\Gamma(\mathcal{E}))$ .

The canonical flat connection is given in any local coordinate system by

$$(17.1.3) \quad \nabla_{\mathrm{can}} = \left( \frac{\partial}{\partial x_\alpha} - \frac{\partial}{\partial \hat{x}} \right) dx_\alpha$$

If a local section of  $\mathcal{E}$  is represented by a vector-valued function  $f(x_\alpha)$ , it defines a horizontal section which is given in local coordinates by  $f(x_\alpha + \hat{x})$ .

**17.2. Real polarization.** Recall that a real polarization is an integrable distribution of Lagrangian subspaces. Let  $\mathcal{P}$  be a real polarization on  $M$ . In this case, automatically  $2c_1(TM) = 0$  modulo 4 (cf. [44]).

**17.2.1. The line bundle  $\mathcal{L}$ .** Assume that  $\omega$  admits a real polarization  $\mathcal{P}$  (i.e. a foliation by Lagrangian submanifolds). By  $T_{\mathcal{P}}$  we denote the quotient of  $TM$  by the subbundle of vectors tangent to the leaves. Choose local Darboux coordinates  $\xi_\alpha, x_\alpha$  such that  $x_{j,\alpha}$  are constant along the leaves. Then the transition coordinate changes are of the form

$$(17.2.1) \quad x_\alpha = g_{\alpha\beta}(x_\beta); \quad \xi_\alpha = (g'_{\alpha\beta}(x_\beta)^t)^{-1}(\xi_\beta + \varphi_{\alpha\beta}(x_\beta))$$

Assume that  $-i\omega$  is a  $2\pi i\mathbb{Z}$ -valued cohomology class. Construct explicitly the line bundle  $\mathcal{L}$  such that  $c_1(\mathcal{L}) = i\omega$ . Adding some constants to  $\varphi_{\alpha\beta}$ , we may assume that  $i\varphi_{\alpha\beta} - i\varphi_{\alpha\gamma} + i\varphi_{\beta\gamma} \in 2\pi i\mathbb{Z}$ ; define  $\mathcal{L}$  to be the line bundle with transition isomorphisms  $\exp(i\varphi_{\alpha\beta})$ . Formulas

$$(17.2.2) \quad A_\alpha = -i\xi_\alpha dx_\alpha$$

define a connection in this bundle, since

$$\xi_\alpha dx_\alpha = \xi_\beta dx_\beta + d\varphi_{\alpha\beta};$$

the curvature of this connection is  $-i\omega$ .

**17.2.2. The jet bundle**  $\text{Jets}(\Gamma_{\text{hor}}(\Omega^{\frac{1}{2}} \otimes \mathcal{L}^k))$ . Define for  $x \in U_\alpha \cap U_\beta$

$$G_{\beta\alpha}(x) : \mathbb{C}[[\hat{x}]] \rightarrow \mathbb{C}[[\hat{x}, \hbar]]$$

by  $(G_{\beta\alpha}f_\alpha)(\hat{x}) = f_\beta(\hat{x})$  where

$$(17.2.3) \quad f_\beta(\hat{x}) = \det g'_{\alpha\beta}(x_\beta + \hat{x})^{\frac{1}{2}} e^{ik\varphi_{\alpha\beta}(x_\beta + \hat{x})} f_\alpha(g_{\alpha\beta}(x_\beta + \hat{x}) - x_\alpha)$$

The square root of the determinant comes from the metalinear structure. The above formula defines the transition functions for the bundle of jets of  $\mathcal{P}$ -horizontal sections of the bundle  $(\wedge^{\max} T_{\mathcal{P}}^*)^{\frac{1}{2}} \otimes \mathcal{L}^k$ .

**17.2.3. The jet bundle**  $\text{Rees Jets } D(\Gamma_{\text{hor}}(\Omega^{\frac{1}{2}} \otimes \mathcal{L}^{\frac{1}{\hbar}}))$ . Recall the construction of the Rees ring and the Rees module [5] of a filtered ring and a filtered module. If  $A$  is a ring with an increasing filtration  $F_p A$ ,  $p \geq 0$ , and  $V$  an  $A$ -module with a compatible filtration  $F_p V$ ,  $p \geq 0$ , we put

$$(17.2.4) \quad \text{Rees } A = \bigoplus_{p \geq 0} \hbar^p F_p A; \quad \text{Rees } V = \bigoplus_{p \geq 0} \hbar^p F_p V.$$

$$(17.2.5) \quad \text{Rees}_f A = \prod_{p \geq 0} \hbar^p F_p A; \quad \text{Rees}_f V = \prod_{p \geq 0} \hbar^p F_p V.$$

When applied to the ring of formal differential operators with its filtration by order, (17.2.4) produces the ring  $\mathbb{C}[[\hat{x}]][\hat{\xi}, \hbar]$  with the usual Heisenberg relations ( $\hat{\xi}_j = i\hbar \frac{\partial}{\partial \hat{x}_j}$ ). When applied to the module of formal functions  $V = \mathbb{C}[[\hat{x}]]$  whose filtration is given by  $F_0 V = V$ , it gives  $\mathbb{C}[[\hat{x}]][\hbar]$ . The completed version (17.2.5) produces the complete Weyl algebra  $\mathbb{C}[[\hat{x}, \hat{\xi}, \hbar]]$  and the complete module  $\mathbb{C}[[\hat{x}, \hbar]]$ .

Observe that in the expression  $G_{\beta\alpha}(i\hbar \frac{\partial}{\partial \hat{x}})G_{\alpha\beta}$  one can substitute  $\frac{1}{i\hbar}$  for  $k$ . The result will be given (in vector/matrix notation) by the following:

$$\frac{1}{2}(i\hbar \frac{\partial}{\partial \hat{x}})(g'_{\alpha\beta}(x_\beta + \hat{x})^t) + (g'_{\alpha\beta}(x_\beta + \hat{x})^t)(i\hbar \frac{\partial}{\partial \hat{x}}) - \varphi'_{\alpha\beta}(x_\beta + \hat{x})$$

Define the bundle of algebras  $\text{Rees Jets } D(\Gamma_{\text{hor}}(\Omega^{\frac{1}{2}} \otimes \mathcal{L}^{\frac{1}{\hbar}}))$  whose fiber is  $\mathbb{C}[[\hat{x}, \hbar]][\hat{\xi}]$  and whose transition isomorphisms are

$$(17.2.6) \quad G_{\beta\alpha}(\hat{x}) = g_{\beta\alpha}(x_\alpha + \hat{x}) - x_\beta;$$

$$(17.2.7) \quad G_{\beta\alpha}(\hat{\xi}) = g'_{\alpha\beta}(x_\beta + \hat{x})^t * \hat{\xi} - \varphi'_{\alpha\beta}(x_\beta + \hat{x})$$

(the multiplication in the left hand side is the (matrix) Moyal-Weyl multiplication). We see that our bundle is the result of formally substituting  $\frac{1}{\hbar}$  for  $k$  in the bundle of jets of Rees rings of  $\mathcal{P}$ -horizontal differential operators on  $(\wedge^{\max} T_{\mathcal{P}}^*)^{\frac{1}{2}} \otimes \mathcal{L}^k$ .

The above formula is the result of formally substituting  $k$  by  $\frac{1}{\hbar}$  into the transition functions for the bundle

$$\text{Rees Jets } D(\Gamma_{\text{hor}}((\wedge^{\max} T_{\mathcal{P}}^*)^{\frac{1}{2}} \otimes \mathcal{L}^k)).$$

**17.2.4. The bundle of algebras  $\widehat{\mathbb{A}}_M$  and the twisted bundle of modules  $\mathcal{H}_M$ .** Now apply to the bundle above the gauge transformation [38]

$$(17.2.8) \quad \text{Ad exp}\left(\frac{1}{i\hbar}\xi_\alpha\widehat{x}\right)$$

We get transition isomorphisms

$$(17.2.9) \quad G_{\beta\alpha}(\widehat{x}) = g_{\beta\alpha}(x_\alpha + \widehat{x}) - x_\beta;$$

$$(17.2.10) \quad G_{\beta\alpha}(\widehat{\xi}) = g'_{\alpha\beta}(x_\beta + \widehat{x})^t * (\widehat{\xi} + \xi_\alpha) - \varphi'_{\alpha\beta}(x_\beta + \widehat{x}) - \xi_\beta$$

Unlike in (17.2.6) and (17.2.7), these transition isomorphisms preserve the maximal ideal  $\langle \widehat{x}, \widehat{\xi}, \hbar \rangle$  and therefore extend to the complete Weyl algebra  $\widehat{\mathbb{A}} = \mathbb{C}[[\widehat{x}, \widehat{\xi}, \hbar]]$ , cf. 2.1. We use them to construct a bundle of algebras  $\widehat{\mathbb{A}}_M$  whose fiber is the Weyl algebra  $\widehat{\mathbb{A}}$ . We see immediately that the bundle of algebras  $\widehat{\mathbb{A}}_M$  is a deformation of the bundle of jets of functions on  $M$ .

Moreover, after we apply the gauge transformation (17.2.8), the formula (17.2.11) allows to replace  $k$  by  $\frac{1}{\hbar}$ . We get new transition isomorphisms

$$(17.2.11) \quad f_\beta(\widehat{x}) = \det g'_{\alpha\beta}(x_\beta + \widehat{x})^{\frac{1}{2}} e^{-\frac{1}{i\hbar}(\varphi_{\alpha\beta}(x_\beta + \widehat{x}) - \varphi'_{\alpha\beta}(x_\beta)\widehat{x})} f_\alpha(g_{\alpha\beta}(x_\beta + \widehat{x}) - x_\alpha)$$

that define a twisted bundle of modules  $\mathcal{H}_M$  whose fiber is the space  $\mathcal{H}$  of the formal metaplectic representation (cf (13.5.1)). The cocycle  $c$  from the definition of a twisted module ((14.1.1)) is  $\exp(\frac{1}{i\hbar}(\varphi_{\alpha\beta} - \varphi_{\alpha\gamma} + \varphi_{\beta\gamma}))$ . (The summand  $-\varphi'_{\alpha\beta}(x_\beta)\widehat{x}$  in the exponent comes from the difference of  $\xi_\alpha\widehat{x}$  and  $\xi_\beta\widehat{x}$  that figure in the gauge transformation).

In other words, the bundle of algebras  $\widehat{\mathbb{A}}_M$  can be formally described as

$$(17.2.12) \quad \widehat{\mathbb{A}}_M = \text{Rees}_f \text{Jets } D_{\text{hor}}((\wedge^{\frac{1}{2}} T_{\mathcal{P}}^*)^{\frac{1}{2}} \otimes \mathcal{L}^{\frac{1}{\hbar}})$$

$$(17.2.13) \quad \mathcal{H}_M = \text{Rees}_f \text{Jets } \Gamma_{\text{hor}}((\wedge^{\frac{1}{2}} T_{\mathcal{P}}^*)^{\frac{1}{2}} \otimes \mathcal{L}^{\frac{1}{\hbar}})$$

(cf. (17.2.5) for the meaning of  $\text{Rees}_f$ ). The latter is only a twisted bundle because the transition functions of  $\mathcal{L}$  stop being a one-cocycle when elevated to the power  $\frac{1}{\hbar}$ .

**17.2.5. The canonical connections.** The bundle of horizontal sections of  $\Omega^{\frac{1}{2}} \otimes \mathcal{L}^k$  has a canonical connection that is given by the formula

$$\nabla = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial \widehat{x}}\right)dx + \frac{\partial}{\partial \xi}d\xi$$

in all local coordinate systems.

This connection induces a connection in  $\text{Jets Rees } D(\Gamma_{\text{hor}}(\Omega^{\frac{1}{2}} \otimes \mathcal{L}^{\frac{1}{i\hbar}}))$  that is given by the same formula. After the gauge transformation from 17.2.4 we get flat connections

$$(17.2.14) \quad \nabla_{\mathbb{A}} = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial \widehat{x}}\right)dx + \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \widehat{\xi}}\right)d\xi$$

in  $\mathcal{A}_M$  and

$$(17.2.15) \quad \nabla_{\mathcal{H}} = -\frac{1}{i\hbar}\xi dx + \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial \widehat{x}}\right)dx + \left(\frac{\partial}{\partial \xi} + \frac{1}{i\hbar}\widehat{x}\right)d\xi$$

**17.3. Complex polarization.** The following is largely based on the approach to deformation quantization from [25].

**17.3.1. Kähler potentials.** Let  $M$  be a Kähler manifold. We can locally choose a Kähler potential, *i.e.* a real-valued function  $\Phi$  such that the symplectic form is given by

$$\omega = -i\partial\bar{\partial}\Phi$$

A Kähler potential is unique up to a change  $\Phi \mapsto \Phi + \varphi + \bar{\varphi}$  where  $\varphi$  is holomorphic.

**Lemma 17.1.** *Put  $\zeta_j = i\frac{\partial\Phi}{\partial z_j}$ . Then*

$$\{z_j, z_k\} = 0; \quad \{\zeta_k, z_j\} = \delta_{jk}; \quad \{\zeta_j, \zeta_k\} = 0.$$

*Proof.* Choose local holomorphic coordinates and put

$$A_{jk} = \frac{\partial}{\partial z_j} \frac{\partial}{\partial \bar{z}_k} \Phi(z, \bar{z})$$

We have

$$\begin{aligned} \{z_j, \bar{z}_k\} &= i(A^{-1})_{kj}; \\ \{z_j, \zeta_k\} &= i \sum \frac{\partial \zeta_k}{\partial \bar{z}_l} \{z_j, \bar{z}_l\} = \sum A_{kl}(A^{-1})_{lj} = \delta_{jk}; \\ -\{\zeta_j, \zeta_k\} &= \sum \left( \frac{\partial \zeta_j}{\partial z_p} \frac{\partial \zeta_k}{\partial \bar{z}_q} - \frac{\partial \zeta_k}{\partial z_p} \frac{\partial \zeta_j}{\partial \bar{z}_q} \right) \{z_p, \bar{z}_q\} = \\ i \sum \left( \frac{\partial^2 \Phi}{\partial z_j \partial z_p} A_{kq} - \frac{\partial^2 \Phi}{\partial z_k \partial z_p} A_{jq} \right) (A^{-1})_{qp} &= i \left( \frac{\partial^2 \Phi}{\partial z_j \partial z_k} - \frac{\partial^2 \Phi}{\partial z_k \partial z_j} \right) = 0 \end{aligned}$$

□

**17.3.2. The line bundle  $\mathcal{L}$ .** Choose an open cover  $\{U_\alpha\}$  of  $M$  and a holomorphic coordinate system  $z_\alpha = (z_{\alpha,1}, \dots, z_{\alpha,n})$  on every  $U_\alpha$ . We write

$$(17.3.1) \quad z_\alpha = g_{\alpha\beta}(z_\beta).$$

Choose local Kähler potentials  $\Phi_\alpha$ . We have

$$(17.3.2) \quad i\Phi_\alpha - i\Phi_\beta = \varphi_{\alpha\beta} + \overline{\varphi_{\alpha\beta}}$$

where  $\varphi_{\alpha\beta}$  are holomorphic.

Let us start with rewriting the transition isomorphisms in terms of the new complex Darboux coordinates  $z, \zeta$ . We have

$$i\Phi_\alpha(z_\alpha) - i\Phi_\beta(z_\beta) = \varphi_{\alpha\beta} + \overline{\varphi_{\alpha\beta}(z_\beta)}$$

Applying  $\frac{\partial}{\partial z_\beta}$ , we get

$$\frac{\partial z_\alpha}{\partial z_\beta} i \frac{\partial \Phi}{\partial z_\alpha}(z_\alpha) - i \frac{\partial \Phi}{\partial z_\beta}(z_\beta) = \frac{\partial \varphi_{\alpha\beta}}{\partial z_\beta}(z_\beta)$$

or

$$(17.3.3) \quad \zeta_\alpha = (g'_{\alpha\beta}(z_\beta)^{-1})^t (\zeta_\beta + \frac{\partial \varphi_{\alpha\beta}}{\partial z_\beta}(z_\beta))$$

Together with (17.3.1), this describes the rule for the change of new variables.

Assume that  $i(\varphi_{\alpha\beta} + \varphi_{\beta\gamma} - \varphi_{\alpha\gamma})$  is a  $2\pi i\mathbb{Z}$ -valued two-cocycle. the line bundle  $\mathcal{L}$  with transition functions  $\exp(\varphi_{\alpha\beta})$ . The curvature of this connection is  $-i\omega$ .

**17.3.3. The jet bundles.** Assume that the canonical sheaf has a square root  $\Omega^{\frac{1}{2}}$ . We call this line bundle the bundle of holomorphic half-forms on  $M$ . The transition isomorphisms of this line bundle are denoted by  $\det g'_{\alpha\beta}{}^{\frac{1}{2}}$ . For any integer  $k$ , consider the bundle  $\text{Jets}(\Gamma_{\text{hol}}(\mathcal{L}^k \otimes \Omega^{\frac{1}{2}}))$  of jets of holomorphic sections of  $\mathcal{L}^k \otimes \Omega^{\frac{1}{2}}$ . The fiber of this bundle is  $\mathbb{C}[[\hat{z}]]$  where  $\hat{z} = (\hat{z}_1, \dots, \hat{z}_n)$ . The transition isomorphisms of the jet bundle take a power series  $f_\alpha(\hat{z})$  to a power series  $f_\beta(\hat{z})$  according to the following formula.

$$(17.3.4) \quad f_\beta(\hat{z}) = f_\alpha(g_{\alpha\beta}(z_\beta + \hat{z}) - z_\alpha) \det g'_{\alpha\beta}(z_\beta + \hat{z})^{\frac{1}{2}} \exp(k\varphi_{\alpha\beta}(z_\beta + \hat{z}))$$

Exactly as in 17.2.3, we can define the bundle of algebras whose fiber is  $\mathbb{C}[[\hat{z}, \hbar]][\hat{\zeta}]$  by transition isomorphisms

$$(17.3.5) \quad G_{\beta\alpha}(\hat{z}) = g_{\beta\alpha}(z_\alpha + \hat{z}) - z_\beta;$$

$$(17.3.6) \quad G_{\beta\alpha}(\hat{\zeta}) = g'_{\alpha\beta}(z_\beta + \hat{z})^t * \hat{\zeta} - \partial_{z_\beta} \varphi_{\alpha\beta}(z_\beta + \hat{z})$$

We see that our bundle is the result of formally substituting  $\frac{1}{\hbar}$  for  $k$  in the bundle of jets of Rees rings of holomorphic differential operators on  $\Omega^{\frac{1}{2}} \otimes \mathcal{L}^k$  (if we map  $\hat{\zeta}_i$  to  $i\hbar\partial_{\hat{z}_i}$ ). On the other hand, because of (17.3.3), this bundle of algebras is a deformation of the bundle of jets of  $C^\infty$  functions on  $M$ . The gauge transformation

$$(17.3.7) \quad \text{Ad exp}\left(\frac{1}{i\hbar}\hat{\zeta}_\alpha\hat{z}\right)$$

produces new transition functions

$$(17.3.8) \quad G_{\beta\alpha}(\hat{z}) = g_{\beta\alpha}(z_\alpha + \hat{z}) - z_\beta;$$

$$(17.3.9) \quad G_{\beta\alpha}(\hat{\zeta}) = g'_{\alpha\beta}(z_\beta + \hat{z})^t * (\hat{\zeta} + \zeta_\alpha) - \partial_{z_\beta} \varphi_{\alpha\beta}(z_\beta + \hat{z}) - \zeta_\beta$$

that extend to  $\hat{\mathbb{A}}_M = \mathbb{C}[[\hat{z}, \hat{\zeta}, \hbar]]$ . The transition isomorphisms for the module of jets (17.3.4) become (when we replace  $k$  by  $\frac{1}{\hbar}$ ) which now define only a



twisted module that we denote by  $\mathcal{H}_M$ .

$$f_\beta(\widehat{z}) = f_\alpha(g_{\alpha\beta}(z_\beta + \widehat{z}) - z_\alpha) \det g'_{\alpha\beta}(z_\beta + \widehat{z})^{\frac{1}{2}} \exp\left(\frac{1}{i\hbar} \varphi_{\alpha\beta}(z_\beta + \widehat{z}) - \partial_{z_\beta} \varphi_{\alpha\beta}(z_\beta) \widehat{z}\right)$$

As in the case of a real polarization, the canonical connections become

$$(17.3.10) \quad \nabla_{\mathbb{A}} = \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \widehat{z}}\right) dz + \left(\frac{\partial}{\partial \zeta} - \frac{\partial}{\partial \widehat{\zeta}}\right) d\zeta$$

in  $\mathcal{A}_M$  and

$$(17.3.11) \quad \nabla_{\mathcal{H}} = -\frac{1}{i\hbar} \zeta dz + \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \widehat{z}}\right) dz + \left(\frac{\partial}{\partial \zeta} + \frac{1}{i\hbar} \widehat{z}\right) d\zeta$$

on  $\mathcal{H}_M$ . We conclude that

$$(17.3.12) \quad \widehat{\mathbb{A}}_M \xrightarrow{\sim} \text{Rees}_f \text{Jets } D_{\text{hol}}(\Omega^{\frac{1}{2}} \otimes \mathcal{L}^{\frac{1}{\hbar}})$$

$$(17.3.13) \quad \mathcal{H}_M \xrightarrow{\sim} \text{Rees}_f \text{Jets } \Gamma_{\text{hol}}(\Omega^{\frac{1}{2}} \otimes \mathcal{L}^{\frac{1}{\hbar}})$$

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